THE BASS–QUILLEN CONJECTURE FOR TORSORS OVER VALUATION RINGS

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ABSTRACT. For a valuation ring V, a smooth V-algebra A, and a reductive V-group scheme G satisfying a certain natural isotropicity condition, we prove that every Nisnevich G-torsor on $\mathbb{A}_A^N$ descends to a G-torsor on A. As a corollary, we generalize Raghunathan's theorem on torsors over affine spaces to a relative setting. We also extend several affine representability results of Asok, Hoyois, and Wendt from equi-characteristics to mixed characteristics. Our proof relies on previous work on the purity of reductive torsors over smooth relative curves and the Grothendieck–Serre conjecture for constant reductive group schemes.

1. INTRODUCTION

Let A be a Noetherian regular ring. The classical Bass–Quillen conjecture, which grew out of Serre's problem that was solved by Quillen in [Qui76] and Suslin in [Sus76], asserts that every vector bundle on the affine space $\mathbb{A}_A^n$ descends to A; see [Qui76, Comment 1 on page 170] and [Bas73, Problem IX]. Building on foundational earlier work, the conjecture was resolved affirmatively in the unramified case or, more generally, when A is regular over a Dedekind ring; see [Qui76], [Lin81], and [Pop89]. Nonetheless, the conjecture remains open in the general case.

In this article, we consider a variant of the conjecture for torsors under reductive group schemes. As will become evident, imposing a specific isotropicity condition on the reductive groups is crucial.

Definition 1.1 (Total isotropicity; see [Čes22a, Definition 8.1] or [Čes22b, Section 1.3.6])**.** Let G be a reductive group scheme over a scheme S. For a point $s \in S$, the adjoint semisimple $\mathscr{O}_{S,s}$-group $G_{\mathscr{O}_{S,s}}^{\mathrm{ad}}$ decomposes as ([SGA 3$_{\mathrm{III\ new}}$, exposé XXIV, proposition 5.10(i)]):

$$G_{\mathscr{O}_{S,s}}^{\mathrm{ad}} \simeq \prod_i \mathrm{Res}_{R_i/\mathscr{O}_{S,s}}(G_i), \tag{1.1.1}$$

where each $\mathrm{Spec}\, R_i \to \mathrm{Spec}\, \mathscr{O}_{S,s}$ is a finite étale cover, and G_i is a simple[1] adjoint R_i-group scheme (of constant type). We say that G is *totally isotropic at* $s \in S$, if each G_i contains a copy of $\mathbb{G}_{m,R_i}$ as an R_i-subgroup. We say that G is *totally isotropic* if it is totally isotropic at each point of S.

Remark 1.2. By [SGA 3$_{\mathrm{III\ new}}$, exposé XXVI, corollaire 6.12], G is totally isotropic at a point $s \in S$ if and only if every G_i in (1.1.1) contains a parabolic R_i-subgroup that is R_i-fibrewise proper.

Examples.

[1]This means that the geometric fibres are reductive algebraic groups with connected Dynkin diagrams.

1.3. A reductive group G over a field k is totally isotropic if and only if every (almost) k-simple factor of its derived subgroup G^{der} is isotropic (i.e., every factor contains a copy of $\mathbb{G}_{m,k}$).

1.4. Tori and quasi-split (in particular, split) reductive group schemes over an arbitrary base scheme are totally isotropic. Also, the total isotropicity is stable under base change and finite products.

Note that Fedorov [Fed21, Def. 1] refers to totally isotropic groups as *strongly locally isotropic* groups.

The vector bundle case $G = \mathrm{GL}_n$ in the following conjecture recovers the original Bass–Quillen conjecture.

Conjecture 1.5 (Bass–Quillen for torsors)**.** *For a Noetherian regular ring A and a* totally isotropic *reductive A-group scheme G, every Nisnevich-locally[2] trivial G-torsor on $\mathbb{A}_A^N$ descends to A, that is,*

$$H^1_{\mathrm{Nis}}(A, G) \xrightarrow{\sim} H^1_{\mathrm{Nis}}(\mathbb{A}_A^N, G) \quad \text{via pullback.} \tag{1.5.1}$$

Remarks.

1.6. Both the total isotropicity of G and the Nisnevich-local triviality (instead of merely étale-local triviality) of torsors are essential: (1) there are non-trivial étale PGL_n-torsors over $\mathbb{A}_k^1$ even for a separably closed but non-algebraically closed field k if $\mathrm{char}(k)|n$, so they do not descend to k (see [CTS21, théorème 5.6.1(vi)]); (2) there exists a generically trivial (in fact, Zariski-locally trivial) $\mathrm{SO}(x_1^2 + \cdots + x_4^2)$-torsor over the affine $\mathbf{R}$-plane $\mathbb{A}_{\mathbf{R}}^2$ that does not descend to $\mathbf{R}$ (see [Par78]). In fact, Balwe and Sawant have shown in [BS17, Theorem 4.8] that if G is defined over an infinite perfect field k and is not totally isotropic, then the pullback map (1.5.1) cannot be bijective for all smooth k-algebras. See also [Fed16] for more concrete counterexamples.

1.7. The Grothendieck–Serre conjecture predicts that an étale torsor under a reductive group scheme G over a regular semilocal domain A is trivial if it trivializes over the fraction field. This conjecture was settled affirmatively when A contains a field ([FP15] and [Pan20]), when A is a semilocal Dedekind domain R (see [Nis82, Nis84] and [Guo22]; this case, by induction, implies the case when A is Henselian). For A that is essentially smooth over R (namely, A is *unramified*), the subcases when G descends to a reductive R-group and when G is totally isotropic are resolved in [GL23, § 1.3] and [ČF23] respectively. For a comprehensive summary of the state of the art, see [GL23] or [ČF23]. The last aforementioned case implies that, when A is unramified in the context of Conjecture 1.5, a G-torsor over $\mathbb{A}_A^N$ is Nisnevich-locally trivial, if and only if it is generically trivial, if and only if it is Zariski-locally trivial. Therefore, Conjecture 1.5 can be reformulated as:

$$H^1_{\mathrm{Zar}}(A, G) \xrightarrow{\sim} H^1_{\mathrm{Zar}}(\mathbb{A}_A^N, G) \quad \text{via pullback.}$$

Apart from the classical vector bundle case $G = \mathrm{GL}_n$, Conjecture 1.5 was proved in the following cases:

- Asok, Hoyois, and Wendt settled the case when A is smooth over a field k and G is the pullback of a totally isotropic reductive k-group; see [AHW18] for k infinite and [AHW20] for k finite.

- Stavrova established the case when A contains a field in [Sta22, Corollary 5.5], and an earlier result [Sta19, Theorem 4.4] addressed the case when A contains an infinite field. (For the convenience of readers, a greatly simplified version of her proof is outlined in § 4.2.)

- Česnavičius, in [Čes22c], independently proved Conjecture 1.5 in the case when A contains a field. This was achieved by completing the solution of Nisnevich's purity conjecture over such an A, as provided by Fedorov [Fed21], and subsequently deriving it from Nisnevich's purity conjecture.

Consequently, the equi-characteristic case of Conjecture 1.5 has been fully settled. However, to the best of our knowledge, the mixed characteristic case remains open. Moreover, as readers will note in § 4.2, the approach used for the equi-characteristic case does not readily extend to the mixed characteristic case at present, primarily because the relative Grothendieck–Serre conjecture (Remark 4.3) remains unresolved.

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[2]Recall that a Nisnevich covering of a scheme X is a family of étale morphisms $\{f_i \colon U_i \to X\}_{i \in I}$ such that for every $x \in X$, there is a $u \in U_i$ lying over x inducing an isomorphism of residue fields $k_x \xrightarrow{\sim} k_u$. In particular, if X is an integral scheme, every Nisnevich cover of X has a generic section. If X is quasi-compact and quasi-separated, then the definition above is equivalent to that the morphism $f \colon \bigsqcup_{i \in I} U_i \to X$ admits a section over each stratum $Z_i \backslash Z_{i+1}$ for a sequence of finitely presented closed subschemes $\varnothing = Z_n \subset Z_{n-1} \subset \cdots \subset Z_1 \subset Z_0 = X$. If X is Noetherian, this follows directly from a Noetherian induction argument. The general quasi-compact quasi-separated case is a little bit subtle and perhaps not so well known; we refer the interested reader to a note of Hoyois, see https://hoyois.app.uni-regensburg.de/papers/allagree.pdf.

The first main result of this article settles a variant of Conjecture 1.5. This result is novel even in the specific case where R is a discrete valuation ring (DVR) of mixed characteristic, stated as follows.

Theorem 1.8 (Theorem 4.1). *Let A be a ring that is ind-smooth over a discrete valuation ring R, and let G be a totally isotropic reductive R-group scheme. Then, via pullback, we have the following bijection*

$$H^1_{\mathrm{Nis}}(A, G) \overset{\sim}{\longrightarrow} H^1_{\mathrm{Nis}}(\mathbb{A}^N_A, G), \quad \text{or, equivalently,} \quad H^1_{\mathrm{Zar}}(A, G) \overset{\sim}{\longrightarrow} H^1_{\mathrm{Zar}}(\mathbb{A}^N_A, G).$$

Moreover, we investigate Conjecture 1.5 for a broad class of rings A that resemble Noetherian regular rings, specifically those that are (ind-)smooth over a Prüfer ring, and thus may be non-Noetherian. Recall that a ring is *Prüfer* if all its local rings are valuation rings; a *valuation ring* is a domain whose ideals are totally ordered by inclusion. While Noetherian Prüfer rings correspond precisely to Dedekind rings, non-Noetherian Prüfer rings are equally prevalent: they arise naturally in non-Archimedean analytic geometry and perfectoid theory, with the integer rings of perfectoid fields serving as notable examples.

More significantly, the recently introduced v- and arc-topologies define covers by testing maps from (spectra of) valuation rings, providing a powerful framework for reducing various problems to analogous ones for valuation rings. Furthermore, growing evidence suggests that a ring smooth over a Prüfer ring exhibits behaviour akin to that of a Noetherian regular local ring; see [GL24] for further discussion of this perspective. These insights motivate the study of geometry over Prüfer rings in *loc. cit.* and, in our scenario, the Bass–Quillen Conjecture 1.5 within this broader framework.

Theorem 1.9 (Theorem 4.1). *Let A be a ring that is ind-smooth over a Prüfer ring R, and let G be a totally isotropic reductive R-group scheme. Then, via pullback, we have the following bijection*

$$H^1_{\mathrm{Nis}}(A, G) \overset{\sim}{\longrightarrow} H^1_{\mathrm{Nis}}(\mathbb{A}^N_A, G), \quad \text{or, equivalently,} \quad H^1_{\mathrm{Zar}}(A, G) \overset{\sim}{\longrightarrow} H^1_{\mathrm{Zar}}(\mathbb{A}^N_A, G).$$

Remark 1.10. In view of this result, we expect that Conjecture 1.5 holds for all totally isotropic reductive group scheme over a ring A (instead of merely over R) that is ind-smooth over a Prüfer ring.

In the case $G = \mathrm{GL}_n$, we note that special subcases of Theorem 1.9 have already been studied in the literature. Specifically, Simis and Vasconcelos considered the case when A is a valuation ring and $N = 1$ in [SV71], while Lequain and Simis treated the case when A is a Prüfer ring in [LS80] (its special case when A is a DVR was previously settled in [Qui76, Theorem 4']). Insofar as we are aware, there are no other instances of Theorem 1.9 in such a non-Noetherian context, even for the simplest group $G = \mathrm{GL}_n$.

The proof of Theorem 1.9 follows the same spirit as the approach to the Bass–Quillen conjecture for vector bundles in [Lin81] and [Pop89], but the technical details are significantly more intricate and demanding. This proof is enabled by recent progress on the mixed characteristic Grothendieck–Serre conjecture obtained in [GL23], particularly the new geometric presentation lemma (Proposition 3.8), and the Prüferian purity result for G-torsors (Theorem 2.4), specifically its important corollary, Theorem 2.7.

1.11. Torsors over relative affine spaces. In [Rag89, Theorem A], Raghunathan proved that for a totally isotropic reductive group G over a field k, a G-torsor over $\mathbb{A}^N_k$ is trivial if it is trivial over $\mathbb{A}^N_{k^s}$ and over the origin $0 \in \mathbb{A}^N_k(k)$, where k^s denotes a separable closure of k. As an application of Theorem 1.9, we can efficiently reprove this result and even present a generalization, as stated below.

Theorem 1.12. *Let A be a domain with fraction field K and G a reductive A-group scheme. Assume*

- *either A is regular Noetherian and contains a field, and G is totally isotropic,*

- *or A is ind-smooth over a Prüfer ring R, and G descends to a totally isotropic reductive R-group.*

Then, a G-torsor over $\mathbb{A}^N_A$ is trivial if and only if it is trivial over $\mathbb{A}^N_{K^s}$ and the zero section $0_A \in \mathbb{A}^N_A(A)$.

Remark 1.13. Assuming the most general case of Conjecture 1.5, our argument will extend Theorem 1.12 to the case of an arbitrary ind-smooth domain A over a Prüfer domain.

1.14. $\mathbb{A}^1$-homotopy theoretic affine representability in mixed characteristics. As applications of Theorem 1.9, we can generalize many results from [AHW18] and [AHW20], extending them beyond equi-characteristics to encompass mixed characteristics as well.

Let S be a quasi-compact quasi-separated scheme. Denote by $\mathrm{Spc}(S)$ the ∞-category of motivic spaces over S (it is overviewed in § 5.1). If $\mathscr{X}$ and $\mathscr{Y}$ are presheaves of spaces on the category of S-smooth

schemes, then we write
$$[\mathscr{X}, \mathscr{Y}]_{\mathbb{A}^1} := \pi_0 \operatorname{Map}_{\operatorname{Spc}(S)}(L_{\mathrm{mot}}(\mathscr{X}), L_{\mathrm{mot}}(\mathscr{Y})),$$
where $L_{\mathrm{mot}} \colon \mathcal{P}(\mathrm{Sm}_S) \to \operatorname{Spc}(S)$ denotes the motivic localization functor (see § 5.1). Given an S-group scheme G, write $\mathbf{B}G$ for its usual bar construction[3], considered as a presheaf of spaces on the category of smooth S-schemes. We establish the following representability of Nisnevich-locally trivial torsors.

Theorem 1.15. *Let S be the spectrum of a Prüfer ring (e.g., a DVR) and let G be a totally isotropic reductive S-group scheme. Then, for every smooth affine S-scheme U, there is a functorial bijection*
$$H^1_{\mathrm{Nis}}(U, G) \simeq [U, \mathbf{B}G]_{\mathbb{A}^1}.$$

In [AHW18, Theorem 4.1.3] and [AHW20, Theorem 2.5], Theorem 1.15 was established under the assumption that S is the spectrum of a field. In fact, using [AHW18, Theorem 2.3.5], Theorem 1.15 can be deduced from the assertion that $\mathbf{B}_{\mathrm{Nis}}G$ is $\mathbb{A}^1$-naive, as defined in [AHW18, Definition 2.1.1] (see § 5.3). This assertion, in turn, is a direct consequence of the Bass–Quillen Theorem 1.9.

As an application of Theorem 1.15, we obtain the following result, which, for certain homogeneous spaces of totally isotropic reductive groups, identifies naive homotopy classes and true $\mathbb{A}^1$-homotopy classes.

Theorem 1.16. *Let S be the spectrum of a Prüfer ring (e.g., a DVR). Suppose that*

(i) *$H \to G$ is a closed immersion of totally isotropic reductive S-group schemes, and*

(ii) *the étale H-torsor $G \to G/H$ is Nisnevich-locally trivial[4].*

Then, G/H is $\mathbb{A}^1$-naive. In particular, for every smooth affine S-scheme U, there is a functorial bijection
$$\pi_0 \left(\operatorname{Sing}^{\mathbb{A}^1} G/H \right) (U) \simeq [U, G/H]_{\mathbb{A}^1}.$$

Theorem 1.16 was previously established in [AHW18] and [AHW20] under the assumption that S is the spectrum of a field.

1.17. Notations and conventions. Throughout this article, we work with commutative rings with units. We freely use the language of ∞-categories, as set out in [HTT] and [HA]. By abuse of notation, the nerve of an ordinary category $\mathcal{C}$ will also be denoted by $\mathcal{C}$, unless otherwise specified.

- For a point s of a scheme (resp., for a prime ideal $\mathfrak{p}$ of a ring), k_s (resp., $k_{\mathfrak{p}}$) denotes its residue field. The total ring of fractions of a ring A is denoted by $\operatorname{Frac} A$. The base change along a morphism of schemes $S \to S'$ is denoted by $(-)_{S'}$; if $S' = \operatorname{Spec} R'$ is affine, we also write $(-)_{R'}$.

- For a quasi-compact quasi-separated scheme S, we let $\mathrm{Sm}_S := \mathrm{Sm}_S^{\mathrm{qcqs}}$ denote the category of quasi-compact quasi-separated smooth S-schemes.

- Let S be a scheme, and let G be an S-group scheme. Given an S-scheme T, a G-torsor over T refers to a $G_T := G \times_S T$-torsor. If G is S-smooth (typically reductive in this article) and $\tau \in \{\mathrm{Nis}, \mathrm{\acute{e}t}\}$, $\mathbf{B}_{\tau} G$ denotes the classifying stack of G-torsors on the (small, big, etc.) τ-site of S.

- $\mathbf{\Delta}$ denotes the simplex category, i.e., the category of finite non-empty linearly ordered sets.

- $\mathcal{S}$ denotes the ∞-category of spaces (or anima, as per [ČS24, § 5.1]).

- For ∞-categories $\mathcal{C}$ and $\mathcal{D}$, let $\mathcal{P}(\mathcal{C}, \mathcal{D}) := \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ denote the ∞-category of $\mathcal{D}$-valued presheaves on $\mathcal{C}$. For brevity, we write $\mathcal{P}(\mathcal{C})$ for $\mathcal{P}(\mathcal{C}, \mathcal{S})$. If τ is a Grothendieck topology on $\mathcal{C}$, we let $\operatorname{Shv}_{\tau}(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ denote the full subcategory of $\mathcal{S}$-valued τ-sheaves.

- $\Delta_{\mathbf{Z}}^{\bullet}$ denotes the standard cosimplicial scheme over $\operatorname{Spec} \mathbf{Z}$, i.e., the functor $\mathbf{\Delta} \to \mathbf{Sch}_{/\mathbf{Z}}$ given by
$$[n] \mapsto \Delta_{\mathbf{Z}}^n := \operatorname{Spec} \left(\frac{\mathbf{Z}[x_0, \cdots, x_n]}{(\sum_{i=0}^n x_i - 1)} \right).$$
For any scheme S, we let $\Delta_S^{\bullet} \colon \mathbf{\Delta} \to \mathbf{Sch}_{/S}$ denote the base change of $\Delta_{\mathbf{Z}}^{\bullet}$ to S.

- $\operatorname{Sing}^{\mathbb{A}^1}$ denotes the singular construction endofunctor on $\mathcal{P}(\mathrm{Sm}_S)$, see (5.1.2).

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[3]Specifically, $\mathbf{B}G$ is the functor that sends an S-smooth scheme U to the nerve of $G(U)$, where $G(U)$ is treated as a category with a single object whose endomorphisms are given by $G(U)$.
[4]Equivalently, generically trivial by [GL23, Theorem 1.1].

2. Purity for torsors under reductive groups

The primary aim of this section is to present the purity of reductive torsors on smooth relative curves over Prüfer bases, as stated in Theorem 2.4. We then derive the main result, Theorem 2.7. Finally, we recall the statement of Grothendieck–Serre conjecture for constant reductive group schemes in Theorem 2.6.

To help the reader get a feeling for working over Prüfer bases, we recall the following ring-theoretic result.

Lemma 2.1. *Let X be a scheme that is flat and locally of finite type over an integral Prüfer scheme S.*

(i) *The scheme X is locally of finite presentation and locally coherent.*

(ii) *For every point $x \in X$, the local ring $\mathscr{O}_{X,x}$ is coherent.*

(iii) *If X is irreducible, then all nonempty S-fibres have the same dimension.*

(iv) *If $\mathscr{O}_{X_s,\xi}$ is reduced for a maximal point $\xi \in X_s$, then the local ring $\mathscr{O}_{X,\xi}$ is a valuation ring and the extension $\mathscr{O}_{S,s} \hookrightarrow \mathscr{O}_{X,\xi}$ induces an isomorphism of value groups.*

Proof. For (i)–(ii), see [GL24, Lemma 3.2.1]. For (iii), see [ÉGA IV$_3$, lemme 14.3.10]. For (iv), see [MB22, théorème A]. $\qquad\square$

The following result, combined with limit arguments, often allows one to only consider Prüfer rings of finite Krull dimension.

Lemma 2.2. *Every semilocal Prüfer domain R is a filtered direct union of its subrings R_i such that:*

(i) *for every i, the ring R_i is a semilocal Prüfer domain of finite Krull dimension; and*

(ii) *for i large enough, $R_i \to R$ induces a bijection on the sets of maximal ideals, hence is fpqc.*

Proof. Write $\mathrm{Frac}(R) = \cup_i K_i$ as the filtered direct union of the subfields of $\mathrm{Frac}(R)$ that are finitely generated over its prime field $\mathfrak{K}$. For $R_i := R \cap K_i$, we have $R = \cup_i R_i$. For (i), it suffices to see that every R_i is a semilocal Prüfer domain whose local rings have finite ranks. Let $\{\mathfrak{p}_j\}_{1 \leqslant j \leqslant n}$ be the set of maximal ideals of R. Then $R = \bigcap_{1 \leqslant j \leqslant n} R_{\mathfrak{p}_j}$ is the intersection of the valuation rings $R_{\mathfrak{p}_j}$. Thus we have

$$R_i = \bigcap_{1 \leqslant j \leqslant n} \left(K_i \cap R_{\mathfrak{p}_j} \right).$$

Since $K_i/\mathfrak{K}$ has finite transcendence degree, by Abhyankar's inequality [Abh56, Corollary 1], every $K_i \cap R_{\mathfrak{p}_j}$ is a valuation ring of finite rank. By [BouAC, VI, § 7, proposition 1–2], R_i is a semilocal Prüfer domain, and its local rings at maximal ideals are precisely the minimal elements of the set $\{K_i \cap R_{\mathfrak{p}_j}\}_{1 \leqslant j \leqslant n}$ under inclusion. This implies (i). For (ii), the quasi-compact property follows from the affineness of $\mathrm{Spec}\, R \to \mathrm{Spec}\, R_i$; each map $R_i \to R$ is flat since R is torsion free. As flat morphism lifts generalizations, it remains to show the bijection between maximal ideals of R_i and of R. Namely, we show that for i large enough there is no strict inclusion relation between $K_i \cap R_{\mathfrak{p}_{j_1}}$ and $K_i \cap R_{\mathfrak{p}_{j_2}}$ for $j_1 \neq j_2$. Indeed, if $\pi_j \in \mathfrak{p}_j \backslash \bigcup_{j' \neq j} \mathfrak{p}_{j'}$ for $1 \leqslant j \leqslant n$, then (ii) holds for any i for which $\{\pi_j\}_{1 \leqslant j \leqslant n} \subset K_i$. $\qquad\square$

The following result will facilitate the passage to the case of closed point in the closed fibre.

Lemma 2.3. *Let V be a valuation ring and let $V \to A$ be an essentially finitely presented (resp., essentially smooth) local homomorphism of local rings. There are an extension of valuation rings $V \subset V'$ identifying their value groups and an essentially finitely presented (resp., essentially smooth) homomorphism $V' \to A$ of V-algebras inducing a finite residue field extension.*

Proof. Assume that $A = \mathscr{O}_{X,x}$ for an affine scheme X finitely presented over V and a point $x \in X$ lying over the closed point $s \in \mathrm{Spec}(V)$. Let $t = \mathrm{tr.deg}(k_x/k_s)$. After shrinking X around x if necessary, choose sections $b_1, \ldots, b_t \in \Gamma(X, \mathscr{O}_X)$ such that their images $\overline{b_1}, \cdots, \overline{b_t}$ in k_x form a transcendental basis of k_x/k_s. Define $p \colon X \to \mathbb{A}_V^t$ by sending the standard coordinates $T_1, \ldots, T_t$ of $\mathbb{A}_V^t$ to $b_1, \ldots, b_t$, respectively. Since the transcendental degree of $k_s(\overline{b_1}, \cdots, \overline{b_t})/k_s$ is t, the image $\eta := p(x)$ is the generic point of $\mathbb{A}_{k_s}^t$, so

$V' := \mathscr{O}_{\mathbb{A}_V^t, \eta}$ is a valuation ring whose value group is $\Gamma_{V'} \simeq \Gamma_V$. Note that k_x/k_η is finite, the map $V' \to A$ induces a finite residue field extension.

Now assume that $V \to A$ is essentially smooth. For a closed point $z \in X$ specializing x, after shrinking X around z, there is a factorization $X \xrightarrow{\pi} \mathbb{A}_V^N \to \operatorname{Spec} V$ for an étale morphism π. As the étale morphism π induces finite separable extension of residue fields, we may assume that $V = k_s$ and $X = \mathbb{A}_s^N$. Since $\overline{\{x\}}$ is an irreducible closed subscheme of $X = \mathbb{A}_s^N$, either we are done or it suffices to take the projection $\mathbb{A}_s^N \to \mathbb{A}_s^{N-1}$ given by a standard coordinate iterately until the image of x is the generic point of $\mathbb{A}_s^t$. $\qquad\square$

The following result provides an analog of the purity theorem of Colliot-Thélène and Sansuc for reductive group torsors over two-dimensional regular Noetherian schemes ([CTS79, théorème 6.13]).

Theorem 2.4 ([GL24, Theorem 6.3]). *Let X be a smooth relative curve over a semilocal affine Prüfer scheme S. Let $Z \subset X$ be a closed subscheme such that the inclusion $j \colon X \backslash Z \hookrightarrow X$ is quasi-compact, and*

$$Z_\eta = \varnothing \quad \text{for each generic point } \eta \in S \quad \text{and} \quad \operatorname{codim}(Z_s, X_s) \geqslant 1 \quad \text{for all } s \in S.$$

Let G be a reductive X-group scheme. Then, restriction induces the following equivalence

$$\mathbf{B}_{\text{ét}} G(X) \xrightarrow{\ \sim\ } \mathbf{B}_{\text{ét}} G(X \backslash Z).$$

In particular, we have a bijection of pointed sets

$$H^1_{\text{ét}}(X, G) \simeq H^1_{\text{ét}}(X \backslash Z, G).$$

Remark 2.5. In higher relative dimensions, the purity Theorem 2.4 is inapplicable, even in the Noetherian setting and for $G = \operatorname{GL}_n$. For instance, for any Noetherian regular local ring $(R, \mathfrak{m}_R)$ of Krull dimension at least 3, there exists a vector bundle over $\operatorname{Spec}(R) \backslash \{\mathfrak{m}_R\}$ that cannot be extended to $\operatorname{Spec} R$.

The following resolution of the Grothendieck–Serre conjecture for constant reductive group schemes is a key input for our Theorem 1.9. In particular, we first use it to deduce Theorem 2.7.

Theorem 2.6 ([GL23, Theorem 1.1]). *For a Prüfer ring R, an irreducible, affine, smooth R-scheme X, and a reductive R-group scheme G, every generically trivial G-torsor on X is Zariski-semilocally trivial:*

$$\text{the sequence} \quad 1 \to H^1_{\text{Zar}}(X, G) \to H^1_{\text{ét}}(X, G) \to H^1_{\text{ét}}(K(X), G) \quad \text{is exact,}$$

where $K(X)$ is the function field of X. In other words, for every semilocal ring A of X, we have

$$\ker\left(H^1_{\text{ét}}(A, G) \to H^1_{\text{ét}}(\operatorname{Frac} A, G)\right) = \{*\}.$$

For any commutative unital ring A, we let $A(t)$ denote the localization $S^{-1}A[t]$ with respect to the multiplicative system S of *monic* polynomials in $A[t]$.

Theorem 2.7. *Let R be a semilocal Prüfer domain and let G be a reductive R-group scheme. Then, a G-torsor over $R(t)$ is trivial if and only if it is generically trivial.*

Proof. Let $\mathcal{E}$ be a generically trivial G-torsor over $R(t)$. Denote by $\mathfrak{r}$ the Jacobson radical of R. We observe that $R(t)$ is the semilocalization of the projective t-line $\mathbb{P}_R^1$ over R along the infinity section $\infty_{R/\mathfrak{r}} \in \mathbb{P}_{R/\mathfrak{r}}^1$, with $s := \frac{1}{t}$ inverted:

$$R(t) = \mathcal{O}_{\mathbb{P}_R^1, \infty_{R/\mathfrak{r}}}\left[\tfrac{1}{s}\right].$$

Note that $\infty_{R/\mathfrak{r}}$ is precisely the set of closed points of $\{s = 0\}$. Hence, $\mathcal{E}$ spreads out to a generically trivial G-torsor $\widetilde{\mathcal{E}}$ on a punctured neighborhood of $\{s = 0\}$ in $\mathbb{P}_R^1$. By patching torsors, we may first extend $\widetilde{\mathcal{E}}$ so that its definition locus contains the generic fibre $\mathbb{P}_K^1$, where $K = \operatorname{Frac}(R)$. Similarly, we may assume that $\widetilde{\mathcal{E}}$ is defined at the maximal point ξ of each R-fibre of $\mathbb{P}_R^1$; note that $\mathcal{O}_{\mathbb{P}_R^1, \xi}$ is a valuation ring. To see this, we may focus on the case where $\operatorname{Spec} R$ (and so $\mathbb{P}_R^1$) is topological Noetherian (using Lemma 2.2); then we again patch torsors using Noetherian induction and the Grothendieck–Serre conjecture for valuation rings ([Guo24, Theorem 1.3]): every generically trivial G-torsor defined on an open subset of $\operatorname{Spec} \mathcal{O}_{\mathbb{P}_R^1, \xi}$ is trivial. Now, the definition locus of $\widetilde{\mathcal{E}}$ satisfies the conditions of $X \backslash Z$ in the purity Theorem 2.4, so $\mathcal{E}$ extends to a G-torsor $\widetilde{\mathcal{E}}$ on $\mathbb{P}_R^1$. Hence, by Theorem 2.6, the pullback of $\widetilde{\mathcal{E}}$ to the semilocal ring $\mathcal{O}_{\mathbb{P}_R^1, \infty_{R/\mathfrak{r}}}$ is trivial, so is the further pullback $\widetilde{\mathcal{E}}|_{R(t)} = \mathcal{E}$. $\qquad\square$

3. TORSORS OVER RELATIVE AFFINE SPACES

3.1. The Quillen patching and its inverse. A central technique for studying torsors over $\mathbb{A}_R^N$ is a local-to-global principle known as Quillen patching. A key insight of Gabber allows one to generalize it to rather general classes of group-valued functors, see [Čes22c, Corollary 5.1.5]. We record it below only for locally finitely presented group schemes, which is the main case of interest for us.

Theorem 3.2. *Let R be a ring and let G be a locally finitely presented R-group scheme.*

- (a) *For a G-torsor X over $R[t_1, \ldots, t_d]$, the set $S \subset R$ of those $r \in R$ such that $X|_{(R[t_1,\ldots,t_d])[\frac{1}{r}]}$ descends to a G-torsor over $R[\frac{1}{r}]$ is an ideal.*

- (b) *A G-torsor over $R[t_1, \ldots, t_d]$ descends to a G-torsor over R iff it does so Zariski-locally on R.*

More generally, the analogues of (a) *and* (b) *hold with $R[t_1, \ldots, t_d]$ replaced by any $\mathbb{Z}_{\geqslant 0}^{\oplus d}$-graded R-algebra $A \cong \bigoplus_{i_1,\ldots,i_d \geqslant 0} A_{i_1,\ldots,i_d}$ such that $R \xrightarrow{\sim} A_{0,\ldots,0}$.*

Compared to Quillen patching Theorem 3.2, the following 'inverse' patching construction is more elementary, but still quite useful. The case where $G = \mathrm{GL}_n$ and $A = R[t_1, \ldots, t_N]$ is due to Roitman [Roi79, Proposition 2].

Lemma 3.3. *Let R be a ring, let G be a quasi-affine, flat, finitely presented R-group scheme, let $A = \bigoplus_{i_1,\ldots,i_N \geqslant 0} A_{i_1,\ldots,i_N}$ be a $\mathbb{Z}_{\geqslant 0}^{\oplus N}$-graded R-algebra (resp., a $\mathbb{Z}_{\geqslant 0}^{\oplus N}$-graded domain over R) such that $R \xrightarrow{\sim} A_{0,\ldots,0}$, and suppose that every G-torsor on A (resp., every generically trivial G-torsor on A) descends to a G-torsor on R. Then, for any multiplicative subset $S \subset R$, every G-torsor on A_S (resp., every generically trivial G-torsor on A_S) whose restriction to each local ring of $(A_{0,\ldots,0})_S \simeq R_S$ extends to a G-torsor on R descends to a G-torsor on R_S.*

(The relevant case for us is when $A = R[t_1, \ldots, t_N]$.)

Proof. We focus on the part on generically trivial torsors, since the other is [Čes22b, Proposition 5.1.10].

Let X be a generically trivial G-torsor on A_S whose restriction to each local ring of $(A_{0,\ldots,0})_S \simeq R_S$ extends to a G-torsor on R. Using Quillen patching Theorem 3.2, we can enlarge S to reduce to the case when R_S is local. Then, by our assumption, the restriction of X to $(A_{0,\ldots,0})_S \simeq R_S$ extends to a G-torsor X_0 on R. Applying a limit argument, we can reduce to the case when $S = r$ is a singleton, at the cost of R_S no longer being local. Notice that the projection onto the $(0, \ldots, 0)$-th component

$$R \oplus \left(\bigoplus_{(i_1,\ldots,i_N) \neq (0,\ldots,0)} A_{i_1,\ldots,i_N}[\tfrac{1}{r}] \right) \simeq A[\tfrac{1}{r}] \times_{R[\frac{1}{r}]} R \twoheadrightarrow R$$

induces an isomorphism both modulo r^n and on r^n-torsion for every $n > 0$. So, by [Čes22b, Proposition 4.2.2], we can glue the G-torsor X on $A[\frac{1}{r}]$ with the G-torsor X_0 on R to obtain a generically trivial G-torsor $\widetilde{X}$ on $A[\frac{1}{r}] \times_{R[\frac{1}{r}]} R$. Observe that

$$A[\tfrac{1}{r}] \times_{R[\frac{1}{r}]} R \simeq \operatorname*{colim}_{i \in \mathbb{N}} A,$$

where the transition maps $A \to A$ are maps of R-algebras that become isomorphisms over R_S and are given by multiplications by $r^{i_1 + \cdots + i_N}$ on the degree $(i_1, \ldots, i_N)$-part $A_{i_1,\ldots,i_N}$. Hence, by a standard limit argument, $\widetilde{X}$ descends to a generically trivial G-torsor on some copy of A in the direct colimit. Therefore, by assumption, it descends further to a G-torsor on R. The base change to R_S of this final descent gives a desired descent of X to a G-torsor on R_S. $\qquad\square$

3.4. Torsors on $\mathbb{A}_R^N$ under reductive R-group schemes. The following result was conjectured in [Čes22b, Conjecture 3.5.1] and settled later in [Čes22c, Theorem 2.1(a)].

Theorem 3.5. *For a ring R and a totally isotropic reductive R-group scheme G, any G-torsor on $\mathbb{A}_R^N$ that is trivial away from some R-finite closed subscheme of $\mathbb{A}_R^N$ is trivial.*

Notice that the isotropicity assumption on G is essential, see, e.g., [Fed16] for counterexamples.

Sketch of proof. Let $\mathcal{P}$ be a G-torsor on $\mathbb{A}_R^N$ that trivializes over $\mathbb{A}_R^N \backslash Z$, where $Z \subset \mathbb{A}_R^N$ is a R-finite closed subscheme. As Z is also finite under the projection $\mathbb{A}_R^N \to \mathbb{A}_R^{N-1}$ onto the first $(N-1)$-coordinates, replacing R with $\mathbb{A}_R^{N-1}$ reduces us to the key case when $N = 1$. In this case, one extends $\mathcal{P}$ to a G-torsor

$\widetilde{\mathcal{P}}$ over $\mathbb{P}^1_R$ (by gluing $\mathcal{P}$ with the trivial torsor over $\mathbb{P}^1_R \backslash Z$). Then $\widetilde{\mathcal{P}}|_{\infty_R}$ is trivial, and, by base change along $\mathbb{A}^1_R \to \operatorname{Spec} R$ (as observed by Gabber), one can reduce to showing that $s^*(\widetilde{\mathcal{P}})$ is trivial for a section $s \in \mathbb{A}^1_R(R)$ (in fact, even $s = 0_R$ by shifting). This statement is insensitive to replacing $\widetilde{\mathcal{P}}$ by its pullback along the map $\mathbb{P}^1_R \to \mathbb{P}^1_R$ given by $[x : y] \mapsto [x^d : y^d]$ for an integer $d > 0$, which allows one to assume that G is moreover semisimple simply connected. Granted these reductions, we revert to prove the stronger statement that $\widetilde{\mathcal{P}}|_{\mathbb{A}^1_R}$ is trivial. For this, Quillen patching Theorem 3.2 reduces us to the case of a local R. In this local case, one can try to modify $\widetilde{\mathcal{P}}$ along ∞_R so that it becomes trivial on the closed fibre $\mathbb{P}^1_{R/\mathfrak{m}_R}$, and then argue using deformation rigidity of the trivial torsor on $\mathbb{P}^1$. $\qquad \square$

For tori, the following result shows that Conjecture 1.5 holds more generally for fppf-torsors (not merely Nisnevich-torsors) over any integral, normal base ring.

Lemma 3.6. *For a normal domain A, an A-group M of multiplicative type, the pullback*

$$H^1_{\mathrm{fppf}}(A, M) \xrightarrow{\sim} H^1_{\mathrm{fppf}}(\mathbb{A}^n_A, M) \quad \text{is bijective.}$$

Proof. When A is Noetherian, this is [CTS87, Lemma 2.4]. For a general normal domain A, we write it as a filtered union of its finitely generated $\mathbf{Z}$-subalgebras A_i, and, by replacing A_i with its normalization (which is again of finite type over $\mathbf{Z}$), we may assume that each A_i is normal. Then we can conclude from the Noetherian case via a limit argument, since M is finitely presented over A. $\qquad \square$

3.7. A Geometric presentation lemma. A useful lemma due to Lindel [Lin81, Proposition 1 and Lemma] states that an étale extension of local rings $A \to B$ with trivial extension of residue fields induces isomorphisms along a well-chosen nonunit $r \in A$:

$$A/r^n A \xrightarrow{\sim} B/r^n B, \quad \text{where} \quad n \geqslant 1.$$

In our context, where the prescribed B is essentially smooth over a valuation ring, we proved a variant of *loc. cit.* that allows us to fix the $r \in B$ in advance, at the cost of the carefully choosing A to be a local ring of an affine space over that valuation ring. This result is one of the crucial geometric tools for attacking the Grothendieck–Serre conjecture for 'constant' reductive group schemes in [GL23]. Like Lindel's work on the Bass–Quillen conjecture for vector bundles, it also reduces the proof of Theorem 1.9 to the case where A is the coordinate ring of an affine open subset of some affine R-space.

For the sake of completeness, we present our Lindel-type result as follows. We will only use its aforementioned special case when $Y = \{r = 0\}$ for a nonunit $r \in A$ and $\#\mathbf{x} = 1$ such that $A/rA \xrightarrow{\sim} B/rB$.

Proposition 3.8 ([GL23, Proposition 4.4])**.** *Let R be a semilocal Prüfer domain. We fix*

- *an irreducible, affine R-smooth scheme X of pure relative dimension $d > 0$,*
- *a finitely presented closed subscheme $Y \subset X$ that is of R-fibrewise codimension > 0, and*
- *a finite subset $\mathbf{x} \subset X$.*

If each fibre of $\mathbf{x}$ over a maximal ideal $\mathfrak{m} \subset R$ has fewer than $\max(\# k_{\mathfrak{m}}, d)$ points, then there are

- *an affine open $W \subset X$ containing $\mathbf{x}$ and an affine open $U \subset \mathbb{A}^d_R$;*
- *an étale R-morphism $f : W \to U$ fitting into the following Cartesian square*

$$\begin{array}{ccc} W \cap Y & \lhook\joinrel\longrightarrow & W \\ \big\| & & \big\downarrow f \\ W \cap Y & \longrightarrow & U, \end{array} \qquad \text{where } f|_{W \cap Y} : W \cap Y \to U \text{ is a closed immersion.}$$

Remark 3.9. The assumption on $\#\mathbf{x}$ holds, for instance, if either $\mathbf{x}$ is a singleton or $d > \#\mathbf{x}$. Note that a certain assumption on $\#\mathbf{x}$ is necessary: when X is a smooth affine curve over a finite field $\Lambda = \mathbb{F}_q$ and $\mathbf{x} \subset X(\mathbb{F}_q)$, the resulting map f from Proposition 3.8 should give an injection $\mathbf{x} \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q}$, which is impossible as soon as $\#\mathbf{x} > q$.

4. Proof of the Bass–Quillen conjecture for torsors over valuation rings

In this section, we prove the following result, the first main result of this paper.

Theorem 4.1. *Let A be a ring that is ind-smooth over a Prüfer ring R, and let G be a totally isotropic reductive R-group scheme. Then, via pullback, we have*

$$H^1_{\mathrm{Nis}}(A, G) \xrightarrow{\sim} H^1_{\mathrm{Nis}}(\mathbb{A}^N_A, G) \quad \text{or, equivalently,} \quad H^1_{\mathrm{Zar}}(A, G) \xrightarrow{\sim} H^1_{\mathrm{Zar}}(\mathbb{A}^N_A, G).$$

4.2. Sketch of proof of Conjecture 1.5 when A contains a field. Before proceeding further, we briefly outline a proof of the equi-characteristic case of Conjecture 1.5, significantly simplifying Stavrova's argument in [Sta22] and also avoiding reliance on Nisnevich's purity conjecture utilized in [Čes22c].

Specifically, by employing Quillen's patching Theorem 3.2 and inducting on N, the problem reduces to showing that, for a totally isotropic reductive group scheme G over a regular local ring A containing a field, every generically trivial G-torsor over $\mathbb{A}^1_A$ is trivial. In other words, we need to show that the following composition map between pointed sets has a trivial kernel:

$$H^1_{\text{ét}}(\mathbb{A}^1_A, G) \xrightarrow{\mu} H^1_{\text{ét}}(\mathbb{A}^1_K, G) \xrightarrow{\nu} H^1_{\text{ét}}(K(t), G),$$

where $K = \mathrm{Frac}(A)$ and t is the standard coordinate on the affine line. But this follows from:

(a) μ has trivial kernel. This follows from the relative form of the Grothendieck–Serre conjecture [Fed22, Theorem 1]: For a regular local ring R containing a field k, with fraction field K, a totally isotropic reductive R-group scheme G, and an affine k-scheme W, no non-trivial G-torsor over $W \times_k R$ trivializes over $W \times_k K$. (It remains to take $W = \mathbb{A}^1_k$.)

(b) ν has trivial kernel. By [Gil02, corollaire 3.10], a generically trivial G-torsor over $\mathbb{A}^1_K$ is isomorphic to $\lambda_*(\mathscr{O}(1))$ for some cocharacter λ of G_K, so it is itself trivial as $\mathscr{O}(1)$ is so. $\qquad\square$

Remark 4.3. The above argument would generalize to the mixed characteristic case if one could prove the relative form of the Grothendieck–Serre conjecture utilized in step (a) in this setting: For a DVR D, an essentially smooth local D-algebra R with fraction field K, a totally isotropic reductive R-group scheme G, and a flat affine D-scheme W, no non-trivial G-torsor over $W \times_D R$ trivializes over $W \times_D K$.

4.4. Proof of Theorem 4.1. Since any section of $\mathbb{A}^N_A \to \mathrm{Spec}\, A$ induces sections to the pullbacks in Theorem 4.1, these pullbacks are injective. Thus, it remains to show that they are surjective. By Quillen patching Theorem 3.2, we may assume that R is a valuation ring by replacing R with its localizations. By a limit argument [Gir71, VII, 2.1.6], it suffices to assume that A is R-smooth. A relative limit argument involving Lemma 2.2 reduces us further to the case of a finite-rank R.

Step 1: *A is a polynomial ring over R.*

It suffices to show that every generically trivial G-torsor $\mathcal{E}$ over $R[t_1, \dots, t_N]$ is trivial; *a fortiori,* it descends to a G-torsor over R. We will argue by double induction on the pair $(N, \mathrm{rank}(R))$. If $N = 0$, then, by convention $R[t_1, \dots, t_N] = R$, the assertion follows from [Guo24, Theorem 1.3] that a generically trivial G-torsor over a valuation ring is trivial. Now assume $N \geqslant 1$ and set $A' := R(t_N)[t_1, \dots, t_{N-1}]$.

Claim 4.4.1. The $G_{A'}$-torsor $\mathcal{E}_{A'}$ descends to a $G_{R(t_N)}$-torsor $\mathcal{E}_0$.

Proof of the claim. Consider the natural projection $\pi \colon \mathrm{Spec}\, R(t_N) \to \mathrm{Spec}\, R$. By definition, $R(t_N)$ is the localization of $R[t_N]$ with respect to the multiplicative system of monic polynomials. Thus, the closed fibre of π is a singleton $\mathfrak{p}_0$. Furthermore, the local ring $R(t_N)_{\mathfrak{p}_0}$ is a valuation ring of $\mathrm{Frac}(R)(t_N)$, and its valuation restricts to the Gauss valuation on $R[t_N]$ associated to R given by

$$R[t_N] \to \Gamma_R, \quad \sum_{i \geqslant 0} a_i t_N^i \mapsto \min_i v(a_i),$$

where $v \colon R \to \Gamma_R$ is the (additive) valuation on R. In particular, we see that R and $R(t_N)_{\mathfrak{p}_0}$ have the same value group. To apply the Quillen patching Theorem 3.2 and conclude, it suffices to show that the base change of $\mathcal{E}_{A'}$ to $R(t_N)_{\mathfrak{p}}[t_1, \dots, t_{N-1}]$ is trivial for every prime ideal $\mathfrak{p} \subset R(t_N)$. If $\mathfrak{p} = \mathfrak{p}_0$, then the above discussion implies that $\mathrm{rank}(R_{\pi(\mathfrak{p})}) = \mathrm{rank}(R)$, so the desired triviality of the base change follows from induction hypothesis. If $\mathfrak{p} \neq \mathfrak{p}_0$, then $\pi(\mathfrak{p}) \in \mathrm{Spec}\, R$ is not the closed point, and we then have $\mathrm{rank}(R_{\pi(\mathfrak{p})}) < \mathrm{rank}(R)$. Therefore, by the induction hypothesis, $\mathcal{E}_{R_{\pi(\mathfrak{p})}[t_1, \dots, t_N]}$ is trivial, and hence its further base change along $R_{\pi(\mathfrak{p})}[t_1, \dots, t_N] \to R(t_N)_{\mathfrak{p}}[t_1, \dots, t_{N-1}]$ is also trivial. $\qquad\square$

Since $\mathcal{E}$ is generically trivial, so is $\mathcal{E}_{A'}$. Recall that the local ring A' at the generic point of the closed fibre over R is a valuation ring, by the Grothendieck–Serre [Guo24, Theorem 1.3], the torsor $\mathcal{E}_{A'}$ is generically trivial on that closed fibre. Hence, by considering the pullback of $\mathcal{E}_{A'}$ along a general section $s \in \mathbb{A}^{N-1}_{R(t_N)}(R(t_N))$, we see that $\mathcal{E}_0$ is also generically trivial because it is trivial at the unique point lying over $\mathfrak{m}_R$. By Lemma 2.7, the torsor $\mathcal{E}_0$, and hence also $\mathcal{E}_{A'}$, is trivial. Consequently, $\mathcal{E}$ is trivial away from the $R[t_1, \ldots, t_{N-1}]$-finite closed subset $\{f = 0\} \subset \mathbb{A}^N_R = \operatorname{Spec} R[t_1, \ldots, t_N]$ for some monic polynomial $f \in R[t_N]$. By Theorem 3.5, the G-torsor $\mathcal{E}$ must be trivial, which completes the induction process.

Step 2: *A is the localization $\widetilde{R}_S$ for a polynomial ring $\widetilde{R} := R[u_1, \ldots, u_d]$ with respect to some multiplicative subset $S \subset \widetilde{R}$.*

We wish to apply the 'inverse' to Quillen patching Lemma 3.3, with $\widetilde{R}$ as the base ring and $\widetilde{R}[t_1, \ldots, t_N]$ as the polynomial ring A. We still need to verify the assumptions there. First, by Step 1, any generically trivial G-torsor over $\widetilde{R}[t_1, \ldots, t_N]$ descends to a G-torsor over $\widetilde{R}$. Second, for any generically trivial G-torsor $\mathcal{E}$ over $\widetilde{R}_S[t_1, \ldots, t_N]$, the restriction of $\mathcal{E}$ to each local ring of the 0-section

$$\operatorname{Spec} \widetilde{R}_S \simeq \{t_1 = \ldots = t_N = 0\} \subset \mathbb{A}^N_{\widetilde{R}_S}$$

is trivial (so extends to the trivial G-torsor over $\widetilde{R}$). Indeed, by Bass–Quillen in the field case, the restriction of $\mathcal{E}$ to $\operatorname{Frac}(\widetilde{R}_S)[t_1, \ldots, t_N]$ is trivial. Thus, the restriction $\mathcal{E}|_{\widetilde{R}_S}$ is generically trivial, and hence is Zariski-locally trivial (Theorem 2.6). This verifies all the assumptions of Lemma 3.3.

Step 3: *A is an arbitrary smooth R-algebra.* Let $\mathcal{E}$ be a generically trivial G-torsor over $\mathbb{A}^N_A$. Our goal is to show that $\mathcal{E}$ descends to a G-torsor over A. Using Quillen patching Theorem 3.2, we may assume that A is an essentially smooth local algebra over a valuation ring R. By localizing R, we can further assume that the ring map $R \to A$ is local.

We will argue by double induction on the pair $(\dim R, \dim A - \dim R)$ to show that $\mathcal{E}$ is even trivial. If $\dim R = 0$, then R is a field, so we conclude from the classical field case settled in § 4.2. If $\dim A = \dim R$, then A is also a valuation ring (Lemma 2.1 (iv)), so we reduce to the case already settled in Step 1. Assume now that $\dim R > 0$ and $\dim A - \dim R > 0$.

By Lemma 2.3, we can enlarge R (without changing $\dim R$) such that A becomes the local ring $\mathcal{O}_{X,x}$, where X is an irreducible affine R-smooth scheme of pure relative dimension $d > 0$, and $x \in X$ is a *closed* point in the *closed* R-fibre of X. Note that $d = \dim A - \dim R$. By our Lindel-type result § 3.7, shrinking X if needed, there are an étale R-morphism $f \colon X \to \mathbb{A}^d_R$ and a nonunit $r_0 \in A_0 := \mathcal{O}_{\mathbb{A}^d_R, f(x)}$ such that

$$f \quad \text{induces a bijection} \qquad A_0/r_0 A_0 \xrightarrow{\sim} A/r_0 A.$$

On the other hand, by induction hypothesis, $\mathcal{E}_{\mathbb{A}^N_{A_\mathfrak{p}}}$ is trivial for any $\mathfrak{p} \in \operatorname{Spec} A \backslash \{\mathfrak{m}_A\}$ (thus descends to the trivial G-torsor over $A_\mathfrak{p}$):

- either $\mathfrak{p}$ lies over a non-maximal ideal $\mathfrak{q} \subset R$, in which case $R_\mathfrak{q} \to A_\mathfrak{p}$ is a local homomorphism,
$$\dim R_\mathfrak{q} < \dim R, \quad \text{and} \quad \dim A_\mathfrak{p} - \dim R_\mathfrak{q} \leqslant d = \dim A - \dim R;$$

- or $R \to A_\mathfrak{p}$ is a local homomorphism, in which case
$$\dim A_\mathfrak{p} - \dim R < \dim A - \dim R.$$

Using Quillen patching Theorem 3.2, we conclude that $\mathcal{E}_{\mathbb{A}^N_{A[1/r_0]}}$ descends to a G-torsor $\mathcal{F}$ over $A[\frac{1}{r_0}]$. Since $\mathcal{F}$ extends to a generically trivial G-torsor over A (for example the restriction of $\mathcal{E}$ along any section $s \in \mathbb{A}^N_A(A)$), it must be trivial by Theorem 2.6. Now, [Čes22a, Lemma 7.1] applies to the Cartesian square

$$
\begin{array}{ccc}
\mathbb{A}^N_{A/r_0 A} & \longhookrightarrow & \mathbb{A}^N_A \\
\downarrow{\scriptstyle \sim} & & \downarrow \\
\mathbb{A}^N_{A_0/r_0 A_0} & \longhookrightarrow & \mathbb{A}^N_{A_0},
\end{array}
$$

we may glue $\mathcal{E}$ with the trivial G-torsor over $\mathbb{A}^N_{A_0[1/r_0]}$ to obtain a G-torsor $\mathcal{E}_0$ over $\mathbb{A}^N_{A_0}$ that trivializes over $\mathbb{A}^N_{A_0[1/r_0]}$. By Step 2, the G-torsor $\mathcal{E}_0$ is trivial, so $\mathcal{E}$ is trivial as well. $\qquad \square$

Building on Theorem 4.1 and its equi-characteristic counterpart settled in § 4.2, we can now derive Theorem 1.12, a generalization of Raghunathan's result, [Rag89, Theorem A], to the relative context.

Proof of Theorem 1.12. In view of Theorem 4.1 and its equi-characteristic counterpart in full generality sketched in § 4.2, it suffices to show the following.

Claim 4.4.2. If (G, A) satisfies Conjecture 1.5, then a G-torsor over $\mathbb{A}_A^N$ is trivial if and only if it is trivial over $\mathbb{A}_{K^s}^N$ and over $0_A \in \mathbb{A}_A^N(A)$, where $K = \operatorname{Frac} A$ and K^s is a separable closure of K.

Proof of the claim. Let $\mathcal{E}$ be a G-torsor over $\mathbb{A}_A^N$ such that $\mathcal{E}|_{\mathbb{A}_{K^s}^N}$ and $\mathcal{E}|_{0_A}$ are both trivial. First, note that it suffices to argue that $\mathcal{E}$ is generically trivial so, in particular, we can assume that A is a field (replacing A by its fraction field): indeed, if so, then $\mathcal{E}$ will descend to A by Conjecture 1.5, and will thus be trivial by checking along the zero section. □

Assume now that $A = K$ is a field (Raghunathan's situation in [Rag89, Theorem A]). We assume by induction that $\mathcal{E}|_{\mathbb{A}_K^{N-1}}$ is already trivial, where $\mathbb{A}_K^{N-1}$ is regarded as a closed subscheme of $\mathbb{A}_K^N$ via

$$\mathbb{A}_K^{N-1} = \mathbb{A}_K^{N-1} \times_K \{0\} \hookrightarrow \mathbb{A}_K^{N-1} \times_K \mathbb{A}_K^1 = \mathbb{A}_K^N.$$

Let $\pi\colon \mathbb{A}_K^N \to \mathbb{A}_K^{N-1}$ denote the projection onto the first $(N-1)$-coordinates, and write $\operatorname{Spec} L$ for the generic point of the target $\mathbb{A}_K^{N-1}$. Then, taking generic fibre of π we obtain a G-torsor $\mathcal{E}|_{\mathbb{A}_L^1}$ which is trivial over $\mathbb{A}_{L^s}^1$ and over $0 \in \mathbb{A}_L^1(L)$. By a result of Raghunathan and Ramanathan (*cf.,* [RR84]), this implies that $\mathcal{E}|_{\mathbb{A}_L^1}$ is trivial, so, in particular, $\mathcal{E}$ is generically trivial, as desired. □

5. $\mathbb{A}^1$-HOMOTOPY THEORETIC AFFINE REPRESENTABILITY IN MIXED CHARACTERISTICS

5.1. The ∞-category of motivic spaces. Let S be a quasi-compact quasi-separated scheme and let Sm_S denote the category of *quasi-compact quasi-separated* (equivalently, finitely presented) S-smooth schemes[5]. The presentable ∞-category $\operatorname{Spc}(S)$ of motivic spaces over S is constructed from the ordinary category Sm_S through the following steps:

a) Formally adjoin all small colimits in the ∞-categorical sense to create $\mathcal{P}(\operatorname{Sm}_S) := \mathcal{P}(\operatorname{Sm}_S, \mathcal{S})$, the ∞-category of presheaves of spaces on Sm_S.

b) Formally invert the class of all Nisnevich covering sieves to create $\operatorname{Shv}_{\operatorname{Nis}}(\operatorname{Sm}_S)$, the ∞-category of Nisnevich sheaves of spaces on Sm_S.

c) Formally contract the affine line $\mathbb{A}_S^1$ by inverting the maps $\operatorname{pr}\colon \mathbb{A}_U^1 \to U$ for all quasi-compact quasi-separated (equivalently, all affine) S-smooth schemes U, to create $\operatorname{Spc}(S)$.

Concretely, $\operatorname{Shv}_{\operatorname{Nis}}(\operatorname{Sm}_S) \subset \mathcal{P}(\operatorname{Sm}_S)$ is the full subcategory spanned by those presheaves of spaces satisfying Nisnevich (Čech) descent. According to Morel–Voevodsky [MV99, Proposition 1.4], a presheaf satisfies Nisnevich descent if and only if it satisfies Nisnevich excision, that is, it sends the empty scheme $\varnothing$ to the final object $* \in \mathcal{S}$ and sends Nisnevich squares to pullback squares in $\mathcal{S}$ (see also [SAG, Appendix B.5.0.3]). As a result, the inclusion $\operatorname{Shv}_{\operatorname{Nis}}(\operatorname{Sm}_S) \subset \mathcal{P}(\operatorname{Sm}_S)$ is stable under both limits and filtered colimits (since filtered colimits commute with pullbacks in $\mathcal{S}$). Furthermore, since the pullback squares form a small set of conditions, this inclusion has a left exact (see [HTT, Lemma 6.2.2.9]), accessible left adjoint (see [HTT, Proposition 5.5.4.15])

$$L_{\operatorname{Nis}}\colon \mathcal{P}(\operatorname{Sm}_S) \to \operatorname{Shv}_{\operatorname{Nis}}(\operatorname{Sm}_S),$$

called the *Nisnevich sheafification* functor. Similarly, by *loc. cit.*, $\operatorname{Spc}(S)$ can be identified with the full subcategory of $\operatorname{Shv}_{\operatorname{Nis}}(\operatorname{Sm}_S)$ spanned by $\mathbb{A}^1$-*local objects*, i.e., those sheaves $\mathscr{F}$ such that $\operatorname{pr}^*\colon \mathscr{F}(U) \xrightarrow{\sim} \mathscr{F}(\mathbb{A}_U^1)$ for all $U \in \operatorname{Sm}_S$; moreover, the inclusion $\operatorname{Spc}(S) \subset \operatorname{Shv}_{\operatorname{Nis}}(\operatorname{Sm}_S)$ admits an accessible left adjoint (which is in general not left exact). Overall, $\operatorname{Spc}(S)$ is a presentable ∞-category, and the inclusion $\operatorname{Spc}(S) \subset \mathcal{P}(\operatorname{Sm}_S)$ admits an accessible left adjoint

$$L_{\operatorname{mot}}\colon \mathcal{P}(\operatorname{Sm}_S) \to \operatorname{Spc}(S),$$

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[5]The quasi-compact quasi-separated assumption has two advantages: 1) the inclusion $\operatorname{Sm}_S \subset \mathbf{Sch}_{/S}$ is stable under taking pullbacks along smooth morphisms; 2) Sm_S is a small category, which cleanly avoids set-theoretical issues, although ultimately this is unnecessary (see Remark 5.2).

called the *motivic localization* functor. Moreover, by construction, the composition

$$\mathrm{Sm}_S \to \mathcal{P}(\mathrm{Sm}_S) \xrightarrow{L_{\mathrm{mot}}} \mathrm{Spc}(S)$$

is a functor such that postcomposing with it induces an equivalence

$$\mathrm{Fun}^L(\mathrm{Spc}(S), \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_S, \mathcal{C})$$

for any ∞-category $\mathcal{C}$ with all small colimits, where $(-)_{\mathrm{Nis},\mathbb{A}^1}$ denotes the full subcategory of functors satisfying Nisnevich codescent and $\mathbb{A}^1$-invariance, and $(-)^L$ denotes the full subcategory of colimit-preserving functors. Finally, by [MV99, § 2, Lemma 3.20], L_{mot} can be described as

$$L_{\mathrm{mot}} = \left(L_{\mathrm{Nis}} \circ \mathrm{Sing}^{\mathbb{A}^1} \right)^{\mathbf{N}} := \mathrm{colim}_{n \to \infty} \underbrace{L_{\mathrm{Nis}} \circ \mathrm{Sing}^{\mathbb{A}^1} \circ \cdots \circ L_{\mathrm{Nis}} \circ \mathrm{Sing}^{\mathbb{A}^1}}_{n \text{ times}}. \tag{5.1.1}$$

Here, the singular construction endofunctor $\mathrm{Sing}^{\mathbb{A}^1} : \mathcal{P}(\mathrm{Sm}_S) \to \mathcal{P}(\mathrm{Sm}_S)$ is described by the formula

$$\mathrm{Sing}^{\mathbb{A}^1} \mathscr{X} := \mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathscr{X}(\Delta_{\mathbf{Z}}^{\bullet} \times_{\mathbf{Z}} -), \tag{5.1.2}$$

where $\Delta_{\mathbf{Z}}^{\bullet}$ denotes the standard cosimplicial scheme over $\mathrm{Spec}\,\mathbf{Z}$. It is known that $\mathrm{Sing}^{\mathbb{A}^1}$ takes values in the full subcategory $\mathcal{P}_{\mathbb{A}^1}(\mathrm{Sm}_S) \subset \mathcal{P}(\mathrm{Sm}_S)$ of $\mathbb{A}^1$-invariant presheaves and in fact it computes the left adjoint to the inclusion $\mathcal{P}_{\mathbb{A}^1}(\mathrm{Sm}_S) \subset \mathcal{P}(\mathrm{Sm}_S)$. In particular, since the category $\mathbf{\Delta}^{\mathrm{op}}$ is sifted [HTT, Lemma 5.5.8.4], it follows that L_{mot} commutes with finite products.

Remark 5.2. In the construction of $\mathrm{Spc}(S)$ above, the category Sm_S can be replaced by either the category of all smooth S-schemes or just the full subcategory $\mathrm{Sm}_S^{\mathrm{aff}}$ of (absolutely) affine ones, without altering the resulting ∞-category $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S)$, and thus $\mathrm{Spc}(S)$, up to equivalences. For instance, to demonstrate that that the restriction functor

$$\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S) \to \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S^{\mathrm{aff}}) \tag{5.2.3}$$

is an equivalence, with its inverse given by the right Kan extension functor, we can apply [HTT, Lemma 6.5.3.9]. This follows from the observation that every scheme admits a bounded[6] Zariski-hypercover where each term is affine: every scheme (resp., separated scheme) has an affine open cover such that every finite intersection is a separated scheme (resp., an affine scheme).

5.3. Representability results. Let $\mathscr{F} \in \mathcal{P}(\mathrm{Sm}_S)$ be a presheaf of spaces on Sm_S. From the construction of $L_{\mathrm{mot}}(\mathscr{F})$ in (5.1.1), there is a canonical morphism

$$\mathrm{Sing}^{\mathbb{A}^1}(\mathscr{F}) \to L_{\mathrm{mot}}(\mathscr{F}). \tag{5.3.4}$$

Following [AHW18, Definition 2.1.1], we say that $\mathscr{F}$ is $\mathbb{A}^1$-*naive* if the restriction of (5.3.4) to $\mathcal{P}(\mathrm{Sm}_S^{\mathrm{aff}})$ is an isomorphism. Under the equivalence (5.2.3), $\mathscr{F}$ is $\mathbb{A}^1$-naive if and only if $\mathrm{Sing}^{\mathbb{A}^1}\mathscr{F} \in \mathrm{Spc}(S)$. This follows, for instance, from the explicit formula (5.1.1) for L_{mot}. In the $\mathbb{A}^1$-naive case, for every $X \in \mathrm{Sm}_S^{\mathrm{aff}}$,

$$\text{the map} \quad \pi_0 \left(\mathrm{Sing}^{\mathbb{A}^1} \mathscr{F} \right)(X) \to \pi_0 \left(L_{\mathrm{mot}}(\mathscr{F})(X) \right) \simeq [X, \mathscr{F}]_{\mathbb{A}^1} \quad \text{is bijective.}$$

The following result is very useful for commuting geometric realizations and pullbacks.

Lemma 5.4. *Let $\mathcal{C}$ be an ∞-category. Consider the following Cartesian diagram in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{P}(\mathcal{C}))$:*

$$\begin{array}{ccc} (\mathscr{X}_3)_{\bullet} & \longrightarrow & (\mathscr{X}_2)_{\bullet} \\ \downarrow & & \downarrow \\ (\mathscr{X}_1)_{\bullet} & \longrightarrow & (\mathscr{X}_0)_{\bullet}. \end{array}$$

If the simplicial object $(\pi_0 \mathscr{X}_0)_{\bullet} \in \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{P}(\mathcal{C}, \mathbf{Set}))$ is constant in the simplicial direction, then the following induced diagram of geometric realizations, valued in $\mathcal{P}(\mathcal{C})$, is also Cartesian:

$$\begin{array}{ccc} |(\mathscr{X}_3)_{\bullet}| & \longrightarrow & |(\mathscr{X}_2)_{\bullet}| \\ \downarrow & & \downarrow \\ |(\mathscr{X}_1)_{\bullet}| & \longrightarrow & |(\mathscr{X}_0)_{\bullet}|. \end{array}$$

---

[6]That is, if $(X_i)_{i=-1}^{\infty}$ denotes a hypercover, then there is an integer N such that if $n > N$, then the canonical map $X_n \to \ker(\prod_{0 \leqslant i \leqslant n} X_{n-1} \rightrightarrows \prod_{0 \leqslant i < j \leqslant n} X_{n-2})$ is an isomorphism.

Proof. Since the problem is sectionwise, we can formally replace $\mathcal{C}$ with the category $*$, allowing us to work with $\mathcal{P}(*) = \mathcal{S}$. As argued in [AHW17, Proof of Lemma 4.2.1], the result then follows from the Bousfield–Friedlander theorem. Alternatively, we could cite [Rez17, Proposition 5.4] or [HA, Lemma 5.5.6.17]. $\square$

Lemma 5.5 (*cf.* [AHW17, Theorem 5.1.3]). *Let*

$$\mathscr{F} \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S) \stackrel{(5.2.3)}{\simeq} \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S^{\mathrm{aff}})$$

be a Nisnevich sheaf. If $\pi_0(\mathscr{F})$ is $\mathbb{A}^1$-invariant on smooth affine S-schemes, then $\mathscr{F}$ is $\mathbb{A}^1$-naive. In particular, for every $U \in \mathrm{Sm}_S^{\mathrm{aff}}$, the natural map

$$\pi_0\big(\mathrm{Sing}^{\mathbb{A}^1}\mathscr{F}\big)(U) \to [U, \mathscr{F}]_{\mathbb{A}^1} \quad \text{is bijective.}$$

Proof. A key result proven in [AHW17, Proposition 2.3.2] (which refines [MV99, Proposition 1.4]) is that a presheaf $\mathscr{G}$ on $\mathrm{Sm}_S^{\mathrm{aff}}$ is a Nisnevich sheaf if and only if it satisfies *affine* Nisnevich excision, i.e., $\mathscr{G}(\varnothing) = *$ and $\mathscr{G}$ sends Nisnevich squares in $\mathrm{Sm}_S^{\mathrm{aff}}$ to pullback squares in $\mathcal{S}$. Given a Nisnevich square

$$\begin{array}{ccc}
V & \hookrightarrow & Y \\
\downarrow & & \downarrow \\
U & \hookrightarrow & X
\end{array}$$

in $\mathrm{Sm}_S^{\mathrm{aff}}$. Then we have the following Cartesian square in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$:

$$\begin{array}{ccc}
\mathscr{F}(X \times_S \Delta_S^{\bullet}) & \longrightarrow & \mathscr{F}(U \times_S \Delta_S^{\bullet}) \\
\downarrow & & \downarrow \\
\mathscr{F}(Y \times_S \Delta_S^{\bullet}) & \longrightarrow & \mathscr{F}(V \times_S \Delta_S^{\bullet}).
\end{array}$$

By assumption, the simplicial set $\pi_0\mathscr{F}(V \times_S \Delta_S^{\bullet})$ is constant, we deduce from Lemma 5.4 (applied to $\mathcal{C} = *$) that the induced diagram of geometric realizations is a Cartesian square in $\mathcal{S}$. By the criterion mentioned above, $\mathrm{Sing}^{\mathbb{A}^1}\mathscr{F}$ is a Nisnevich sheaf on $\mathrm{Sm}_S^{\mathrm{aff}}$. The result then follows. $\square$

We can now finish the proof of Theorems 1.15 and 1.16.

Proof of Theorem 1.15. Let S be the spectrum of a Prüfer ring. Let $\mathbf{B}_{\mathrm{Nis}}G$ denote $L_{\mathrm{Nis}}(\mathbf{B}G)$, the Nisnevich sheafification of $\mathbf{B}G$. Then for every $U \in \mathrm{Sm}_S$, $(\mathbf{B}_{\mathrm{Nis}}G)(U)$ is (the nerve of) the groupoid of Nisnevich-locally trivial G-torsors over U, so that $\pi_0(\mathbf{B}_{\mathrm{Nis}}G)(U) = H^1_{\mathrm{Nis}}(U, G)$. By the Bass–Quillen Theorem 4.1, the presheaf $H^1_{\mathrm{Nis}}(-, G)$ on $\mathrm{Sm}_S^{\mathrm{aff}}$ is $\mathbb{A}^1$-invariant, so Lemma 5.5 implies that $\mathrm{Sing}^{\mathbb{A}^1}(\mathbf{B}_{\mathrm{Nis}}G) \simeq L_{\mathrm{mot}}(\mathbf{B}_{\mathrm{Nis}}G)$. To conclude, note that since $L_{\mathrm{mot}}(\mathbf{B}G) \simeq L_{\mathrm{mot}}(\mathbf{B}_{\mathrm{Nis}}G)$, by Lemma 5.5 we have:

$$\begin{aligned}
[U, \mathbf{B}G]_{\mathbb{A}^1} &= \pi_0 \mathrm{Map}_{\mathrm{Spc}(S)}(L_{\mathrm{mot}}(U), L_{\mathrm{mot}}(\mathbf{B}G)) \\
&\simeq \pi_0 \mathrm{Map}_{\mathrm{Spc}(S)}(L_{\mathrm{mot}}(U), L_{\mathrm{mot}}(\mathbf{B}_{\mathrm{Nis}}G)) \\
&= [U, \mathbf{B}_{\mathrm{Nis}}G]_{\mathbb{A}^1}. \quad \square
\end{aligned}$$

Proof of Theorem 1.16. Let $* = S \to \mathbf{B}_{\mathrm{Nis}}G$ denote the base point classifying the trivial G-torsor over S. We then have the following Cartesian diagram in $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S)$:

$$\begin{array}{ccc}
G/_{\mathrm{Nis}}H & \longrightarrow & \mathbf{B}_{\mathrm{Nis}}H \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathbf{B}_{\mathrm{Nis}}G,
\end{array} \qquad (5.5.5)$$

where the right vertical arrow is induced by the inclusion $H \subset G$ via functoriality, and $G/_{\mathrm{Nis}}H$ denotes the Nisnevich sheafification of $\mathrm{Sm}_S \ni U \mapsto G(U)/H(U)$. By assumption, we can identify $G/_{\mathrm{Nis}}H$ with the usual fpqc quotient G/H restricted to Sm_S. Evaluating the diagram (5.5.6) at $\Delta_S^{\bullet} \times_S -$, we obtain a Cartesian diagram in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{P}(\mathrm{Sm}_S^{\mathrm{aff}}))$. Since the reductive S-group G is totally isotropic, Theorem 4.1 implies that the simplicial object

$$\pi_0(\mathbf{B}_{\mathrm{Nis}}G)(\Delta_S^{\bullet} \times_S -) = H^1_{\mathrm{Nis}}(\Delta_S^{\bullet} \times_S -, G) \in \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{P}(\mathrm{Sm}_S^{\mathrm{aff}}, \mathbf{Set}))$$

is constant in the simplicial direction. By taking geometric realizations, Lemma 5.4 yields the following Cartesian diagram in $\mathcal{P}(\mathrm{Sm}_S^{\mathrm{aff}})$:

$$(5.5.6)\qquad\begin{array}{ccc}
\mathrm{Sing}^{\mathbb{A}^1}(G/H) & \longrightarrow & \mathrm{Sing}^{\mathbb{A}^1}(\mathbf{B}_{\mathrm{Nis}}H) \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathrm{Sing}^{\mathbb{A}^1}(\mathbf{B}_{\mathrm{Nis}}G).
\end{array}$$

By the Bass–Quillen Theorem 4.1 and the $\mathbb{A}^1$-naive criterion Lemma 5.5, we have $\mathrm{Sing}^{\mathbb{A}^1}(\mathbf{B}_{\mathrm{Nis}}H)$, $\mathrm{Sing}^{\mathbb{A}^1}(\mathbf{B}_{\mathrm{Nis}}G) \in \mathrm{Spc}(S)$. As the inclusion $\mathrm{Spc}(S) \subset \mathcal{P}(\mathrm{Sm}_S^{\mathrm{aff}})$ is stable under all limits, it follows that $\mathrm{Sing}^{\mathbb{A}^1}(G/H) \in \mathrm{Spc}(S)$, i.e., G/H is $\mathbb{A}^1$-naive. This completes the proof. $\square$

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