哈爾濱工業大學
HARBIN INSTITUTE OF TECHNOLOGY

# Purity for flat cohomology for syntomic algebras over valuation rings

Ning Guo

IASM-HIT

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# Outline of the talk

1. From Purity to Solving Equations
2. Valuation Rings and Arc Descent
3. Purity for Flat Cohomology over Valuation Rings

# Purity and Hartogs Extension

## From Hartogs to purity

Hartogs' extension phenomenon comes from several complex variables.
In algebraic geometry it reappears as **purity**: geometric objects and invariants should be insensitive to removing closed subsets of sufficiently high codimension.

If $n \geq 2$, holomorphic functions on a punctured polydisc extend uniquely.

$$\mathcal{O}(\Delta^n) \simeq \mathcal{O}(\Delta^n \setminus \{0\}).$$

# Typical Purity Phenomena

- Line bundles on a normal scheme extend uniquely across closed subsets of codimension $\geq 2$.
- Vector bundles extend across codimension-2 closed subsets (codimension 3 already has counterexamples: there are vector bundles that cannot extend along $\mathbb{A}^3 \backslash \{0\} \hookrightarrow \mathbb{A}^3$)
- Torsors under reductive group schemes extend across codimension-2 closed subsets.
- Let $X$ be regular and let $Z \subset X$ be closed of codimension $\geq 2$. Brauer purity gives

$$\mathrm{Br}(X) \simeq \mathrm{Br}(X \setminus Z).$$

# Purity for the Brauer Group

## Purity statement

Let $X$ be a regular scheme and $Z \subset X$ be closed of codimension $\geq 2$. We have

$$\mathrm{Br}(X) \simeq \mathrm{Br}(X \setminus Z).$$

- Brauer classes are controlled by codimension-1 divisors.
- Higher-codimension points contribute no additional residues.

# The Brauer Purity Conjecture: A Short History

- Auslander–Goldman: an algebraic purity via commutative rings and Azumaya algebras.[1]
- Grothendieck promoted the Brauer group to the étale cohomological Brauer group and predicted purity for regular schemes.[2]
- de Jong recorded Gabber's Azumaya–cohomological comparison theorem.[3]
- Thomason–Gabber absolute purity, through the Kummer sequence, handles the prime-to-residue-characteristic torsion.[4]
- Česnavičius completed the mixed-characteristic $p$-torsion case using perfectoid methods.[5]

*Historical references*

[1] Auslander–Goldman, *The Brauer group of a commutative ring* (1960).
[2] Grothendieck, *Le groupe de Brauer* I–III (1964–66).
[3] A. J. de Jong, *A result of Gabber* (2003), Thm. 1.1: $\mathrm{Br}(X) = \mathrm{Br}'(X)$ under the ample line bundle hypothesis.
[4] R. W. Thomason, *Absolute cohomological purity*, Bull. Soc. Math. France 112 (1984), 397–406; K. Fujiwara, Gabber's absolute purity, Adv. Stud. Pure Math. 36 (2002), 153–183.
[5] Česnavičius, *Purity for the Brauer group* (Duke Math. J. 2019).

# Purity for the Brauer Group and Arithmetic

Number theory asks whether polynomial equations over a number field $k$ have solutions or integral solutions. Geometrically, for a $k$-variety $X$, one asks

$$X(k) \neq \emptyset? \qquad \text{and more generally, how can one describe or compute } X(k)?$$

The first test is local: for every place $v$ of $k$, check whether

$$X(k_v) \neq \emptyset.$$

If one local equation has no solution, then no global solution exists.

**The local-to-global question**

If $X(k_v) \neq \emptyset$ for every $v$, must $X(k)$ be nonempty?

This is the Hasse principle; its refined version is weak approximation.

# The Brauer–Manin Obstruction

Manin's idea is to cut out a more refined adelic subset using the Brauer group:

$$X(\mathbb{A}_k)^{\mathrm{Br}} = \left\{ (P_v) \in X(\mathbb{A}_k) \mid \sum_v \mathrm{inv}_v\big(\mathcal{A}(P_v)\big) = 0 \text{ for all } \mathcal{A} \in \mathrm{Br}(X) \right\}.$$

Here $\mathrm{inv}_v\big(\mathcal{A}(P_v)\big) \in \mathbb{Q}/\mathbb{Z}$ is the local invariant obtained from the local point $P_v$. Global points always satisfy the reciprocity constraint:

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k).$$

Thus, if

$$X(\mathbb{A}_k) \neq \emptyset \quad \text{but} \quad X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset,$$

then $X$ has points everywhere locally but has no global point. This is a **Brauer–Manin obstruction**.

# Why Purity Makes the Brauer Group Computable

For a regular integral scheme $X$, one often constructs Brauer classes over the function field $K = k(X)$, while $\mathrm{Br}(X)$ itself is harder to compute. Purity says that Brauer classes on $X$ are exactly those function-field classes that are unramified along every divisor. For an integer $n$ invertible on $X$, one has the residue exact sequence

$$0 \longrightarrow \mathrm{Br}(X)[n] \longrightarrow \mathrm{Br}(K)[n] \xrightarrow{\oplus \partial_D} \bigoplus_{D \in X^{(1)}} H^1(\kappa(D), \mathbb{Z}/n\mathbb{Z}).$$

Equivalently,

$$\mathrm{Br}(X)[n] = \{\alpha \in \mathrm{Br}(K)[n] \mid \partial_D(\alpha) = 0 \text{ for all divisors } D\}.$$

This turns the Brauer–Manin set into something that can be computed using residues.

# Outline

1. From Purity to Solving Equations
2. Valuation Rings and Arc Descent
3. Purity for Flat Cohomology over Valuation Rings

## Definition (valuation rings)

A subring $V \subset K$ of a field is a **valuation ring** if for every $x \in K$ one has either $x \in V$ or $x^{-1} \in V$.

- The prime spectrum Spec $V$ is totally ordered by inclusion.
- A valuation may be written as
$$v : K \longrightarrow \Gamma \cup \{\infty\},$$
where $\Gamma$ is a totally ordered abelian group and
$$v(0) = \infty, \quad v(1) = 0, \quad v(ab) = v(a) + v(b), \quad v(a+b) \geq \min\{v(a), v(b)\}.$$
- The rank of the valuation ring is its Krull dimension:
$$\mathrm{rank}(V) := \dim V.$$

# Examples of Valuation Rings

- Discrete valuation rings. In fact, a Noetherian valuation ring is either a field or a DVR.
- Rings of integers of perfectoid fields, such as $\mathcal{O}_{\mathbb{C}_p}$; these are typically non-Noetherian.
- Higher-rank valuation rings, e.g. $k[[u]] + v\,k((u))[[v]] \subset k((u))((v))$, a rank-2 valuation ring with value group $\mathbb{Z}^2_{\mathrm{lex}}$.
- Local rings of $\overline{\mathbb{Z}}$, the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$.
- Classical Riemann–Zariski space of $K/k$: a point is a valuation ring $V \subset K$ over $k$, equivalently a compatible choice of centers on proper models.[1]

1 Zariski–Samuel, *CA II*, Ch. VI, §17; Temkin, *Relative Riemann–Zariski spaces*, Intro.

## Toward Arc Descent

Arc topology: valuation rings of rank $\leq 1$ test covers; arc descent glues along those tests.

# The Arc Topology

Let $f : Y \to X$ be a qcqs morphism of schemes.

**Arc cover (Bhatt–Mathew, *The arc-topology*, 2021)**

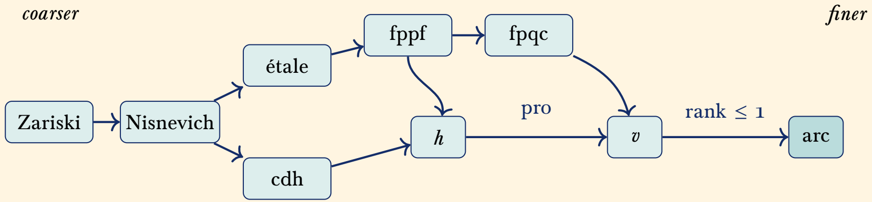
The morphism $f$ is an **arc cover** if, for every valuation ring $V$ of rank $\leq 1$ and every map

$$\operatorname{Spec} V \longrightarrow X,$$

there exist a rank $\geq 1$ extension of valuation rings $V \to W$ and a lift making the diagram commute:

$$\begin{tikzcd} \operatorname{Spec} W \arrow[r, dashed, "\exists"] \arrow[d] & Y \arrow[d, "f"] \\ \operatorname{Spec} V \arrow[r] & X. \end{tikzcd}$$

# Arc Topology Among Familiar Topologies




- cdh = Nisnevich + abstract blow-ups (proper modifications, isomorphisms off closed subsets).
- $h$ = fppf covers + proper surjective morphisms (loc. f. pr.).
- fpqc $\Rightarrow$ $v$: flat going-up + domination by a valuation ring.

# Arc Descent

Let

$$F : \mathrm{Sch}_{\mathrm{qcqs}}^{\mathrm{op}} \longrightarrow C$$

be a contravariant functor valued in an ∞-category or a stable category.

## Definition

The functor $F$ satisfies **arc descent** if, for every arc cover $Y \to X$, there is an equivalence

$$F(X) \xrightarrow{\sim} \mathrm{Tot}\big(F(Y^{\bullet/X})\big)$$

Thus $F(X)$ can be reconstructed from the Čech nerve of any arc cover $Y \to X$.

Objects satisfying arc descent can be checked, computed, and glued over valuation rings.

# Typical Examples of Arc Descent

- Bhatt–Mathew: for a finite ring $\Lambda$, the functor $X \mapsto D_c^b(X,\Lambda)$ satisfies arc descent; in particular, $X \mapsto R\Gamma(X_{\text{ét}},\Lambda)$ is an arc sheaf.
- Bhatt–Mathew: $\mathrm{Perf}(X)$[1] on qcqs perfect schemes in characteristic $p$ satisfies arc descent.
- Bhatt–Scholze: arc covers of $p$-adic formal schemes are tested by $p$-complete rank-1 valuation rings and faithfully flat rank-1 extensions. Perfectoid formal schemes form a basis for this topology.
- Every $p$-complete ring is covered by $p$-complete perfectoid rank-1 valuation rings.
- Kazuhiro Ito: finite projective modules and Perf[1] over perfectoid rings satisfy $p$-complete arc descent.

1 $\mathrm{Perf}(-)$ denotes the stable $\infty$-category of perfect complexes.

# Outline

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# Main Theorem

**Theorem (G., 2026)**

*Let $V$ be a valuation ring with $\operatorname{Spec} V \setminus \{\mathfrak{m}_V\}$ quasi-compact. For $x$ closed in the closed fiber of $X/V$, with $X/V$ of relative dimension $d$, set $A = \mathcal{O}_{X,x}$ and $\mathfrak{m} = \mathfrak{m}_A$. Let $G$ be a finite commutative $A$-group scheme.*

(1) *If $G$ is flat, then*

$$H^i_{\mathfrak{m}}(A, G) = 0 \quad \begin{cases} i < d+1, & A/V \text{ is ind-syntomic,} \\ i \leq d+1, & A/V \text{ is ind-smooth.} \end{cases}$$

(2) *If $A/V$ is ind-smooth and $G$ is étale of invertible order, then $H^i_{\mathfrak{m}}(A, G) = 0$ for $i < 2d+2$.*

(3) *If $A/V$ is ind-smooth, $R := A^{\mathrm{sh}}$, $U_R := \operatorname{Spec} R \setminus \{\mathfrak{m}_R\}$, and $d \geq 1$, then $H^2(U_R, \mathbb{G}_m) = 0$.*

# From Local Cohomology to Brauer Purity

**(1) + (2) $\Longrightarrow$ (3).**

(2) Prime-to-$p$ part: by the stronger étale vanishing,

$$H^3_{\mathfrak{m}}(A, G) = 0$$

in relative dimension 1, since $3 < 2 \cdot 1 + 2$.

(1) $p$-primary part, $d \geq 2$: the ind-smooth case of the flat-cohomology vanishing gives

$$H^3_{\mathfrak{m}}(A, G) = 0.$$

- $p$-primary part, $d = 1$: the closed point has weak dimension exactly 2. By the vector-bundle purity theorem of Guo–Liu[1], Azumaya algebras extend uniquely across the closed point, and purity follows.

1 Guo–Liu, *Purity and quasi-split torsors over Prüfer bases*, J. Éc. polytech. Math. 11 (2024), 817–877.

# Preliminary Reductions

- $V = \text{filtered}\,\mathrm{colim}_\lambda V_\lambda$ for finite dim. valuation rings $V_\lambda$; hence assume $\dim V < \infty$.
- $U_A$ is quasi-compact. Choose a finitely generated ideal $I \subset A$ with $\sqrt{I} = \mathfrak{m}_A$; depth can then be measured with respect to $I$.
- First split the coefficients into primary factors:

$$G \simeq \prod_{\ell \mid |G|} G_\ell, \qquad |G_\ell| = \ell^{n_\ell}.$$

# Coefficient Reduction by Primary Components

Since $A$ is local, primary decomposition gives

$$G \simeq \prod_{\ell \mid |G|} G_\ell, \qquad |G_\ell| = \ell^{n_\ell}.$$

**Prime-to-$p$ branch**
If $\ell \in A^\times$, then $G_\ell$ is finite étale. After strict Henselization it is constant, and a composition series reduces to

$$\mathbb{Z}/\ell\mathbb{Z}.$$

**$p$-primary branch**
If $\ell = p = \mathrm{char}(\kappa(x))$, $G_p$ need not become constant. Keep $G_p$ finite flat and use its Dieudonné/prismatic module.

**Berthelot–Breen–Messing input**. Locally on the base, embed $G \hookrightarrow H$ into a truncated $p$-divisible group; let $Q = H/G$ be finite flat.

**Česnavičius–Scholze / this proof**. The exact sequence $0 \to G \to H \to Q \to 0$, together with the known $i < d$ vanishing, reduces the top degree from $G$ to $H$. The filtration $H[p] \subset H[p^2] \subset \cdots \subset H$ then reduces to $pH = 0$.

*References:* Berthelot–Breen–Messing, *Théorie de Dieudonné cristalline II*, Th. 3.1.1; Česnavičius–Scholze, proof of Th. 6.2.7.

# Difficulties in the Prime-to-$p$ Case

The classical Noetherian strategy does not directly extend to valuation rings.

- The proof in the quasi-excellent setting uses Gabber's cohomological results; these are unavailable in the present non-Noetherian situation.
- Gabber's argument over a DVR uses the Hoobler sequence. For imperfect valuation rings this has essential difficulties, and a reduction to the perfection is not formally sound.
- One needs Verdier duality over valuation rings, especially a dualizing complex that sees the boundary at the closed fiber.

## Key tools

- Logarithmic dualizing complexes over valuation rings
- Hansen–Scholze relative perverse $t$-structures
- Affine Artin vanishing

# The Dualizing Complex over a Valuation Ring

Gabber introduced a dualizing complex over valuation rings to capture information at the closed boundary. Let $W$ be a rank-1 valuation ring, $T = \operatorname{Spec} W$, with closed point and generic point

$$i : s \hookrightarrow T, \qquad j : \eta \hookrightarrow T.$$

Assume the residue characteristic is $p > 0$, and let $\ell \neq p$ be a prime. For the value group $\Gamma$, set

$$r_\ell := \dim_{\mathbb{F}_\ell} \Gamma/\ell\Gamma < \infty.$$

The **dualizing complex** is

$$\omega_{T,\ell} := \tau^{s}_{\le r_\ell - 1} Rj_* \mathbb{Z}_\ell,$$

where $\tau^{s}_{\le r_\ell - 1}$ means truncation only at the closed point:

$$i^*(\tau^{s}_{\le r_\ell - 1} K) = \tau_{\le r_\ell - 1} i^* K, \qquad j^*(\tau^{s}_{\le r_\ell - 1} K) = j^* K \quad (K = Rj_* \mathbb{Z}_\ell).$$

After coefficient dévissage, one may pass to finite $\ell$-primary coefficients.

# Logarithmic Interpretation

**Theorem (G., 2026)**

*Let* $\widehat{\Gamma}_\ell := \varprojlim_n \Gamma/\ell^n\Gamma$. *The pro-$\ell$ log tame inertia is*

$$I^{\log}_{T,\ell} := \varprojlim_n D_s(\ell^{-n}\Gamma/\Gamma) \simeq \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}_\ell}(\widehat{\Gamma}_\ell, \mathbb{Z}_\ell(1)_s).$$

*Thus, noncanonically,* $I^{\log}_{T,\ell} \simeq \mathbb{Z}_\ell(1)^{r_\ell}_s$. *If* $\mathcal{Z}$ *is the reduced closed fiber of the* $\ell^\infty$*-root stack*

$$\sqrt[\ell^\infty]{T} \longrightarrow T,$$

*then* $\mathcal{Z} \simeq [C^{\log}_\Gamma / D(\ell^{-\infty}\Gamma)]$ *is an* $I^{\log}_{T,\ell}$*-banded gerbe over* $s$. *If* $T$ *has a global log chart, then* $\mathcal{Z} \simeq \mathbf{B} I^{\log}_{T,\ell}$.

# Orientation Class

## Theorem (G., 2026)

*If $p : \mathcal{Z} \to s$ is the structure map, then*

$$Rp_*\mathbb{Z}_\ell \simeq R\psi\mathbb{Z}_\ell = i^* Rj_*\mathbb{Z}_\ell.$$

*Moreover, Gabber's dualizing complex is*

$$\omega_{T,\ell} := \mathrm{fib}\big(Rj_*\mathbb{Z}_\ell \to i_* \det(R^1 p_*\mathbb{Z}_\ell)[-r_\ell]\big), \qquad \omega_{T,\ell} \simeq \tau^{s}_{\leq r_\ell - 1} Rj_*\mathbb{Z}_\ell.$$

# Orientation Class (continued)

**Orientation via localization**

Let $E \to B$ be an oriented real rank-$r$ vector bundle, $i : B \hookrightarrow E$, and $j : U = E \setminus B \hookrightarrow E$. The fibre orientation gives the Thom term

$$i^! \Lambda_E \simeq \Lambda_B[-r].$$

After tensoring by the base orientation line $L_B$, localization gives

$$i_* L_B[-r] \longrightarrow L_E \longrightarrow Rj_* L_U \xrightarrow{\partial_\phi} i_* L_B[1-r].$$

Thus $L_E$ is obtained by gluing the base line $L_B$ with the oriented fibre Thom class.

**Valuative boundary**

The boundary gerbe $p : \mathcal{Z} \to s$ supplies the analogue of the fibre orientation line

$$L_{T,\ell} := \det\left(R^1 p_* \mathbb{Z}_\ell\right).$$

The shifted object $i_* L_{T,\ell}[-r_\ell]$ is the Thom/top-boundary term; in a residue triangle it appears with the extra shift $+1$. Gabber's complex is the fibre of the truncation to this oriented top term:

$$\omega_{T,\ell} = \mathrm{fib}\left(Rj_* \mathbb{Z}_\ell \to i_* L_{T,\ell}[-r_\ell]\right).$$

This is the valuative analogue of recovering the oriented total space from the punctured part plus the boundary orientation class.

# Hansen–Scholze Relative Perversity

**Theorem (Hansen–Scholze, 2023)**

*Let $f : X \to S$ be a morphism of finite presentation. On $D(X)$ there is a **relative perverse t-structure***

$$ {}^{p/S}D^{\leq 0}, \qquad {}^{p/S}D^{\geq 0}, $$

*defined fiberwise by the usual perverse t-structure, and $\mathrm{Perv}(X/S)$ is an abelian category.*

# Relative Artin Vanishing

**Theorem (Hansen–Scholze, 2023)**

*If $V$ is an absolutely integrally closed rank-1 valuation ring and $a : Y \to X$ is affine of finite presentation, then*

$$Ra_* : {}^{p/S}D^{\leq 0}(Y) \longrightarrow {}^{p/S}D^{\leq 0}(X)$$

*is right t-exact.*

This is the **relative Artin vanishing** theorem.

# AIC Valuation Rings and Verdier Duality

Let $V$ be an absolutely integrally closed rank-1 valuation ring and let $p : X \to S$ be smooth of relative dimension $d$. The dualizing complex on $X$ takes the form

$$\omega_\ell \simeq j_{X!}\Lambda_{X_\eta}(d)[2d+1].$$

Define the $\ell$-adic Verdier duality functor

$$\mathbb{D}_{X,\ell}(-) := R\mathcal{H}om_X(-,\omega_\ell).$$

It satisfies $\mathbb{D}^2_{X,\ell} \simeq \mathrm{id}$ and is compatible with the six operations. For example, for $f : Y \to X$ separated of finite presentation between $S$-smooth schemes,

$$\mathbb{D}_X Rf_! \simeq Rf_*\mathbb{D}_Y, \qquad \mathbb{D}_X Rf_* \simeq Rf_!\mathbb{D}_Y.$$

For an open immersion $v : Y \hookrightarrow X$,

$$\mathbb{D}_X v_! \simeq Rv_*\mathbb{D}_Y.$$

# Closed Points and Duality

Let

$$i_x : x \to X_s \hookrightarrow X$$

be a geometric closed point. One has

$$i_x^! \omega_\ell \simeq \Lambda(d)[2d],$$

and therefore

$$i_x^! F \simeq R\operatorname{Hom}_\Lambda\left(i_x^* \mathbb{D}_X F, \Lambda(d)[2d]\right).$$

This converts lower bounds for local cohomology into upper bounds after Verdier duality.

# Affine Lefschetz Along Arcs

## Affine Lefschetz hyperplane theorem

Let $V$ be a finite-dimensional valuation ring, and let $A$ be an ind-smooth local $V$-algebra with maximal ideal $\mathfrak{m}$. For any $f \in \mathfrak{m}$ and any invertible prime $\ell$, the map

$$H^i_{\mathfrak{m}}(A, \mathbb{Z}/\ell\mathbb{Z}) \longrightarrow H^i_{\mathfrak{m}}(A/f, \mathbb{Z}/\ell\mathbb{Z})$$

is an isomorphism for $i < d$ and injective for $i = d$.

This supplies the Lefschetz input for the prime-to-$p$ case after arc descent.

# The Local Form of Lefschetz

Using the coniveau spectral sequence

$$E_1^{a,b} = H_x^{a+b}(i_x^! F) \Longrightarrow H^{a+b}_{\{f=0\}}(X, F),$$

one reduces to the following local statement. Let

$$u : X[1/f] \hookrightarrow X$$

be an open immersion, where $X$ is smooth over $V$, and let $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$. For every geometric point $x \to X$ with $f(x) = 0$, prove

$$R\Gamma_x(X_x, u_!\Lambda) \in D^{\geq d+1}(\Lambda),$$

or equivalently

$$i_x^! u_! \Lambda \in D^{\geq d+1}(\Lambda).$$

# Proof Idea via AIC Arc Descent

- AIC arc descent: for $F \in D_c^b(X,\Lambda)$ and a geometric point $x \to X$, the condition

$$R\Gamma_x(i_x^! F) \in D^{\geq N}$$

is local for the arc topology. Thus it suffices to check rank-1 AIC valuation rings in an arc hypercover.

- Put

$$M_U := j_{U!}\Lambda_{U_\eta}(d)[d] \in D_c^b(U,\Lambda).$$

This is perverse.

# Duality Step

- By Hansen–Scholze relative Artin vanishing,

$$Ru_*M_U \in {}^{p/S}D^{\leq 0}, \qquad \text{so} \qquad i_x^*Ru_*M_U \in D^{\leq 0}.$$

- By six-functor duality,

$$\mathbb{D}_X(u_!\Lambda[d]) \simeq Ru_*M_U[1].$$

Hence $i_x^*\mathbb{D}_X(u_!\Lambda[d]) \in D^{\leq -1}$, and duality gives

$$i_x^! u_!\Lambda \in D^{\geq d+1}(\Lambda).$$

# Roadmap for $p$-Primary Groups

**Characteristic $p$**
**injective reductions**

**Perfectize the base**
$$A \longrightarrow A \otimes_V V^{\mathrm{perf}}$$
syntomic Čech descent

**Perfectize the ambient algebra**
$$B \longrightarrow B^{\mathrm{perf}}$$
descent along ind-syntomic maps

**Adjoin $p$-power roots**
$$P/(f_i) \longrightarrow P/(f_i^{1/p^\infty})$$
root-adjunction comparison

**Mixed characteristic**
**equivalence and limit**

**Complete along the support**
$$A \longrightarrow \widehat{A}$$
excision for derived completion

**Choose a perfectoid cover**
$$\widehat{A} \longrightarrow B$$
André's lemma

**Totalize completed terms**
$$\{\widehat{B^{[m]}}\}_{m\geq 0}$$
$p$-complete flat descent

**Reduced to perfectoid purity**

# Perfectization Steps

## Step 1: perfectize the valuation ring

Adjoin all $p$-power roots to $V$. The perfectization

$$V \longrightarrow V^{\mathrm{perf}}$$

is ind-syntomic, so Čech descent allows one to assume that $V$ is perfect.

# Perfectization Steps

## Step 2: reduce to a perfect ambient smooth algebra

If

$$A = B/(f_1, \ldots, f_c)$$

with $B$ smooth over a perfect valuation ring $V$, then

$$B \longrightarrow B^{\mathrm{perf}}$$

is a faithfully flat ind-syntomic extension. Descent reduces the problem to

$$B^{\mathrm{perf}}/(f_1, \ldots, f_c).$$

# Adding $p$-Power Roots to the lci Equations

In a perfect ambient ring $P$, replace

$$R = P/(f_1, \dots, f_c)$$

by

$$R_\infty = P/(f_1^{1/p^\infty}, \dots, f_c^{1/p^\infty}).$$

The elements $a_1, \dots, a_d$ defining the support remain a regular sequence in $R_\infty$. The comparison theorem of Česnavičius–Scholze gives

$$H_I^i(R, G) \longrightarrow H_I^i(R_\infty, G)$$

injective for $i < d$ and an isomorphism for $i < d - 1$. Thus it remains to prove, for the characteristic-$p$ perfectoid ring $R_\infty$,

$$H_I^q(R_\infty, G) = 0 \qquad (q < d).$$

# André Covers with Compatible Roots

## Definition used here

For chosen equations $f_1, \ldots, f_c \in A_0$, an **André cover** means a map

$$A_0 \longrightarrow B \quad \text{faithfully flat ind-syntomic}, \qquad \widehat{B} \text{ perfectoid}, \qquad f_i^{1/p^\infty} \in \widehat{B}.$$

Here $f_i^{1/p^\infty}$ means a compatible sequence

$$(f_i^{1/p^{n+1}})^p = f_i^{1/p^n}, \quad (f_i^{1/p})^p = f_i.$$

It is an existence statement for a good cover, not a canonical formula $B = A_0[f_i^{1/p^\infty}]$.

**Use in the proof.** For $R = \widetilde{R}/(f_1, \ldots, f_c)$, pass to the Čech nerve of $A_0 \to B$. After completion, each term is perfectoid and contains $f_i^{1/p^\infty}$; then apply perfectoid purity to the quotients by the root towers $(f_i^{1/p^\infty})$.

Reference: Česnavičius–Scholze, *Purity for flat cohomology*, §2.3, especially Theorem 2.3.4, and the use in §6.2.

# Perfectoid Purity via Prismatic Dieudonné Modules

Let $R$ be a perfectoid ring and set

$$A_{\mathrm{inf}} := W(R^\flat), \qquad \xi = \ker(\theta : A_{\mathrm{inf}} \to R).$$

A finite flat $p$-primary group scheme $G$ corresponds to a prismatic Dieudonné module $\mathbf{M}_G$, of projective dimension at most 1 over $A_{\mathrm{inf}}$, equipped with Frobenius and Verschiebung. The key comparison is

$$R\Gamma_I(R, G) \simeq R\Gamma_{IA_{\mathrm{inf}}}(A_{\mathrm{inf}}, \mathbf{M}_G)^{V=1}.$$

The problem is reduced to coherent local cohomology, and the depth assumptions give low-degree vanishing.

# $p$-Primary Reductions: Summary

## Characteristic $p$: injective reductions

$$A_1 = A \otimes_V V^{\mathrm{perf}}, \qquad A_2 = B^{\mathrm{perf}}/(f_1, \dots, f_c), \qquad R_\infty = P/(f_i^{1/p^\infty}).$$

$$\begin{aligned} H^q_I(A, G) &\hookrightarrow H^q_{IA_1}(A_1, G) \\ &\hookrightarrow H^q_{IA_2}(A_2, G) \\ &\hookrightarrow H^q_{IR_\infty}(R_\infty, G) = 0. \end{aligned}$$

- first two injections: syntomic Čech descent plus lower-degree induction;
- third injection: root-adjunction comparison;
- final vanishing: perfectoid flat purity with supports.

## Mixed characteristic: descent equivalences

After derived $I$-completion, André gives a perfectoid cover whose completed Čech terms are $\widehat{B^{[m]}}$.

$$\begin{aligned} R\Gamma_I(A, G) &\simeq R\Gamma_I(\widehat{A}, G) \\ &\simeq R\lim_{\underset{m}{\longleftarrow}} R\Gamma_{\widehat{IB^{[m]}}}(\widehat{B^{[m]}}, G) \\ &\in D^{\geq d}. \end{aligned}$$

- first equivalence: excision for derived $I$-completion;
- second equivalence: $p$-complete arc hyperdescent;
- last line: termwise perfectoid purity, and $D^{\geq d}$ is closed under limits.

**Perfectoid local input: support purity; in the top smooth degree, the prismatic $V = 1$ argument.**

# Thank you

Ning Guo

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# The Top Degree: Čech Representatives

In the smooth perfectoid case, the support is cut out by a regular sequence $a_1, \dots, a_d$, and

$$\xi, a_1, \dots, a_d$$

remains regular in $A_{\inf}$. Thus the only degree not killed by depth is the top Čech term:

$$H^d_{IA_{\inf}}(A_{\inf}, \mathbf{M}_G) \cong \frac{\mathbf{M}_G[a_1^{-1}, \dots, a_d^{-1}]}{\sum_j \mathbf{M}_G[a_1^{-1}, \dots, \widehat{a_j^{-1}}, \dots, a_d^{-1}]}.$$

At the perfectized stage, compatible $p$-power roots of the $a_i$ are available. A top class is represented by finite sums of pole monomials

$$m\, a_1^{-\lambda_1} \cdots a_d^{-\lambda_d}, \qquad \lambda_i \in \mathbb{Z}[1/p]_{>0}.$$

# The Top Degree and Verschiebung

The Verschiebung on $\mathbf{M}_G$ is $\varphi^{-1}$-linear. On the leading Čech pole, it acts by pulling exponents back through Frobenius:

$$m\, a_1^{-\lambda_1} \cdots a_d^{-\lambda_d} \quad \xmapsto{V} \quad V(m)\, a_1^{-\lambda_1/p} \cdots a_d^{-\lambda_d/p}.$$

For a nonzero class $x$, choose a maximal pole vector $\mathrm{pole}(x) = (\lambda_1, \ldots, \lambda_d)$ among the finitely many monomials in a Čech representative. Then

$$\mathrm{pole}(Vx) \leq \frac{1}{p}\,\mathrm{pole}(x) < \mathrm{pole}(x).$$

Thus no nonzero top-degree class can satisfy $Vx = x$:

$$H^d_{IA_{\mathrm{inf}}}(A_{\mathrm{inf}}, \mathbf{M}_G)^{V=1} = 0, \qquad H^d_I(R, G) = 0.$$

Together with the lower-degree depth vanishing, this gives in the smooth case

$$H^i_I(A, G) = 0 \qquad (i \leq d).$$